RECOGNIZING MAPPING SPACES

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ABSTRACT. Given a fixed object A in a suitable pointed simplicial model category $\mathcal{C}$, we study the problem of recovering the target Y from the pointed mapping space $\operatorname{map}_*(A, Y)$ (up to A-equivalence). We describe a recognition principle, modelled on the classical ones for loop spaces, but using the more general notion of an *A-mapping algebra.* It has an associated transfinite procedure for recovering $\operatorname{CW}_A Y$ from $\operatorname{map}_*(A, Y)$, inspired by Dror-Farjoun's construction of CW_A-approximations.

INTRODUCTION

Given two topological spaces A and Y, the set $\operatorname{Hom}_{\mathcal{T}}(A, Y)$ of all continuous maps from A to Y can be endowed with the compact-open topology, forming the space $\underline{\operatorname{Hom}}(A, Y)$ of maps from A to Y. Up to homotopy, this may be replaced by the simplicial set $\operatorname{map}(A, Y)$ (with $\operatorname{map}(A, Y)_n := \operatorname{Hom}_{\mathcal{T}}(A \otimes \Delta[n], Y)$, where $\Delta[n]$ is the standard simplicial n-simplex). Mapping spaces, pointed and unpointed, play a central role in homotopy theory, and two basic questions arising in their study are:

(a) Given a space A, when is X of the form $\operatorname{map}(A, Y)$ for some space Y (up to weakly equivalence)?
(b) If X satisfies the conditions prescribed in the answer to (a), how can we recover Y from $X \simeq \operatorname{map}(A, Y)$?

In this paper we concentrate on *pointed* mapping spaces, since these appear to be more useful in applications. The special case when $A = S^n$ has been studied intensively since the 1950s – see, e.g., [Su1, Su2, Sta1, May, Sm, C, Ba]. Less has been done on the general problem (but see [BD, Bl, CDI, FT, IS]).

0.1. *Remark.* Note that the idea of recovering Y itself from $X := \operatorname{map}_*(A, Y)$ does not really make sense: this is evident even for $A = S^1$, where $\Omega Y = \operatorname{map}_*(A, Y)$ depends only on the base-point component of Y. In fact, ΩY is weakly equivalent to $\Omega(\operatorname{CW} Y)$, where $\operatorname{CW} Y$ is the usual CW-approximation of Y. More generally, for any $A, Y \in \mathcal{T}_*$ we have $\operatorname{map}_*(A, Y) \simeq \operatorname{map}_*(A, \operatorname{CW}_A Y)$, where $\operatorname{CW}_A Y$ is the A-CW approximation of Y (cf. [DF, §2]). Thus the best we can hope for is to recover Y from $\operatorname{map}_*(A, Y)$ *up to A-equivalence* (see Definition 1.6 below).

0.2. **Summary of results.** Our approach is based on the notion of an *A-mapping algebra,* first introduced in [BB1]): this encodes the action of mapping spaces between various colimits constructed out of A on the corresponding limits of $\operatorname{map}_*(A, Y)$.

It can be thought of as a continuous version of Lawvere's concept of an algebra over a theory Θ, replacing the latter by a simplicially enriched category Θ_A, with $\mathfrak{X} : \Theta_A^{\mathrm{op}} \to \mathcal{S}_*$ a continuous product-preserving functor.

In fact, one can define the notion of an extended A-mapping algebra by adding certain (homotopy) colimits to Θ_A (to obtain a suitable enriched sketch $\Theta_{\mathcal{F}}$, in the sense of Ehresmann – cf. [E]), and requiring $\mathfrak{X}$ to take these colimits to limits in $\mathcal{S}_*$ (see Section 1).

A judicious choice of the colimits we allow provides enough extra structure on $\mathrm{map}_*(A, -)$ to be able to recognize the image of this functor. The technical apparatus needed to specify these choices, using the Stover construction of [Sto], is given in Section 2, with the connection to mapping algebras explained in Section 3.

In Section 4 we explain how to use this extra structure on X to inductively construct a transfinite sequence of objects $Y^{(\alpha)}$ as appropriate homotopy colimits, and show:

Theorem A. *Any pointed simplicial set X equipped with an extended A-mapping algebra structure $\hat{\mathfrak{X}}$ is weakly equivalent to $\mathrm{map}_*(A, \mathrm{hocolim}_{\alpha<\lambda} Y^{(\alpha)})$, where $(Y^{(\alpha)})_{\alpha<\lambda}$ is the sequence of objects constructed in Section 4.*

See Theorem 5.1 below. This provides a theoretical answer to the first question, too – although, as with many recognition principles, it is essentially impossible to verify in practice.

There is also a homotopy-invariant version of the second approach, in which we replace A-mapping algebras by their lax version (see §6.2), using an appropriate cofibrant replacement of Θ_A, described in Section 6. We then show:

Theorem B. *If X is a pointed simplicial set equipped with a lax extended A-mapping algebra structure, and the sequence $(\widetilde{Y}^{(\alpha)})_{\alpha<\lambda}$ is constructed in §6.15, then $X \simeq \mathrm{map}_*(A, Y)$, for $Y := \mathrm{hocolim}_{\alpha<\lambda} \widetilde{Y}^{(\alpha)}$. Moreover, if $X \simeq X'$, then X' also has an induced lax extended A-mapping algebra structure, from which we can recover Y' (A-equivalent to Y) with $X' \simeq \mathrm{map}(A, Y')$. .*

See Theorem 6.19 below.

In §6.21 we observe that when A is a wedge of (possibly localized) spheres of various dimensions, we can use a lax version of Stover's construction (see [Sto]) to recover Y up to homotopy from $X \simeq \mathrm{map}_*(A, Y)$. This allows one to bypass the anomalous behavior of the usual "homotopy multiplicative" loop space recognition and recovery machinery described in [BD].

Finally, in the Appendix we sketch an alternative "Dold-Lashof" approach to recovering the target of a mapping space, in the form of an obstruction theory based on the construction of [DF, §2B] (see Theorem 7.10).

0.3**. Notation and conventions.** The category of compactly generated Hausdorff spaces (cf. [Ste], and compare [V]) is denoted by $\mathcal{T}$, and that of pointed connected compactly generated spaces by $\mathcal{T}_*$. For any category $\mathcal{C}$, $s\mathcal{C} := \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ is the category of simplicial objects over $\mathcal{C}$. We abbreviate $s\mathcal{S}et$ to $\mathcal{S}$, $s\mathcal{S}et_*$ to $\mathcal{S}_*$, and $s\mathcal{G}p$ to $\mathcal{G}$. The full subcategory of fibrant objects (Kan complexes) in $\mathcal{S}_*$ is denoted by $\mathcal{S}_*^{\mathrm{Kan}}$.

If $\langle \mathcal{V}, \otimes \rangle$ is a monoidal category, we denote by $\mathcal{V}\text{-}\mathcal{C}at$ the collection of all (not necessarily small) categories enriched over $\mathcal{V}$ (see [Bor, §6.2]). For any set $\mathcal{O}$, denote by $\mathcal{O}\text{-}\mathcal{C}at$ the category of all small categories $\mathcal{D}$ with $\mathrm{Obj}\,\mathcal{D} = \mathcal{O}$. A $(\mathcal{V}, \mathcal{O})$-*category* is a category $\mathcal{D} \in \mathcal{O}\text{-}\mathcal{C}at$ enriched over $\mathcal{V}$, with $\mathcal{V}$-mapping objects $\mathrm{map}_{\mathcal{D}}^{v}(-, -)$. The category of all $(\mathcal{V}, \mathcal{O})$-categories will be denoted by $(\mathcal{V}, \mathcal{O})\text{-}\mathcal{C}at$.

The main examples of $\langle \mathcal{V}, \otimes \rangle$ we have in mind are $\langle \mathcal{S}et, \times \rangle$, $\langle \mathcal{G}p, \times \rangle$, $\langle \mathcal{G}pd, \times \rangle$, $\langle \mathcal{S}, \times \rangle$, and $\langle \mathcal{S}_{*}, \wedge \rangle$.

0.4**. Assumptions.** Throughout this paper we consider only pointed simplicial model categories $\mathcal{C}$: This means that in addition to the pointed simplicial mapping space $\mathrm{map}_{*}(X, Y) = \mathrm{map}_{\mathcal{C}}(X, Y) \in \mathcal{S}_{*}$ for any $X, Y \in \mathcal{C}$, we have two bifunctors $- \otimes - : \mathcal{C} \times \mathcal{S} \to \mathcal{C}$ and $(-)^{(-)} : \mathcal{C} \times \mathcal{S}^{\mathrm{op}} \to \mathcal{C}$, as in [Q, II, §1-2].

We may assume that the n-simplices of $\mathrm{map}_{*}(X, Y)$ are given by $\mathrm{map}_{*}(X, Y)_{n} := \mathrm{Hom}_{\mathcal{C}}(X \otimes \Delta[n], Y)$, where $\Delta[n]$ is the standard simplicial n-simplex in $\mathcal{S}$.

We assume further that $\mathcal{C}$ has all limits and colimits, is cofibrantly generated and left proper (cf. [Hi, §13.1], with functorial factorizations (cf. [Ho, §1.1.1]), such that all objects are fibrant (e.g., $\mathcal{C} = \mathcal{T}_{*}$ or $\mathcal{G}$). We make an additional assumption on the simplicial structure in §2.2 below.

When this does not hold – for example, for $\mathcal{C} = \mathcal{S}_{*}$ – one can sometimes use an appropriate Dwyer-Kan equivalence $R : \mathcal{C} \to \mathcal{D}$ to a category $\mathcal{D}$ with all objects fibrant to produce a new enrichment $\mathrm{map}'_{\mathcal{C}}(A, Y) := \mathrm{map}_{\mathcal{D}}(RA, RY)$ whose properties we can then study by the methods described here. We shall not pursue this idea further in this paper.

1. Mapping algebras

Any mapping space $X = \mathrm{map}_{*}(A, Y) \in \mathcal{S}_{*}$ (for fixed A) has additional structure, given by the action of $\mathrm{map}_{*}(A, A)$ on X, and so on. This can be useful in recovering Y from X, as we know from the example of $A = S^{1}$ (see [Su1, Sta1]). In order to codify this extra structure, we shall use the following version of the notion of a *mapping algebra,* introduced in [BB1, §8] (see also [BB2, §2]).

Recall that for a pointed simplicial model category $\mathcal{C}$ as in §0.4, one can define the n-fold suspension $\Sigma^{n} A$ of any $A \in \mathcal{C}$ to be $A \otimes \Delta[n]/A \otimes \partial \Delta[n]$ (see [Q, I, §2] and compare [DKSt1, §5.2]). We let $\Sigma A := \Sigma^{1} A$.

1.1. **Definition.** Let A be a fixed cofibrant object in $\mathcal{C}$, and let Θ_{A} be the full sub-simplicial category of $\mathcal{C}$ generated by A under suspensions and coproducts of cardinality $< \kappa$, for a suitably chosen cardinal κ (see §4.1 below). More generally, we can let $\mathcal{F}$ be any set of cofibrant objects in $\mathcal{C}$, with $\Theta_{\mathcal{F}}$ the full sub-simplicial category of $\mathcal{C}$ generated by $\mathcal{F}$ under suspensions and coproducts of cardinality $< \kappa$. A simplicial functor $\mathfrak{X} : \Theta_{\mathcal{F}}^{\mathrm{op}} \to \mathcal{S}_{*}^{\mathrm{Kan}}$, written $B \mapsto \mathfrak{X}\{B\}$, is called an $\mathcal{F}$-*mapping algebra* if $\mathfrak{X}$ takes coproducts in Θ_{A} to products and suspensions to loops:

(1.2) $$\mathfrak{X}\{\coprod_{i} B_{i}\} \cong \prod_{i} \mathfrak{X}\{B_{i}\} \quad \text{and} \quad \mathfrak{X}\{\Sigma B\} \cong \Omega \mathfrak{X}\{B\}$$

for $B, B_i \in \mathcal{F}$. Note that each $\mathfrak{X}\{B\}$ is fibrant, by assumption. The category of $\mathcal{F}$-mapping algebras will be denoted by $\mathcal{M}ap_{\mathcal{F}}$. In particular, when $\mathcal{F} = \{A\}$, an $\mathcal{F}$-mapping algebra is called an *A-mapping algebra,* and the category of A-mapping algebras will be denoted by $\mathcal{M}ap_A$.

1.3. *Remark.* One could rephrase (1.2) more succinctly by requiring that $\mathfrak{X}$ preserve the loops and products of the *opposite* category $\mathbf{\Theta}_{\mathcal{F}}^{\mathrm{op}} \subseteq \mathcal{C}^{\mathrm{op}}$. However, since the main example we have in mind is $\mathcal{C} = \mathcal{T}_*$, we thought it would be easier for the reader to keep track of the constructions in the original category $\mathcal{C}$.

1.4. **Definition.** For each $Y \in \mathcal{C}$ we have a *realizable* $\mathcal{F}$-mapping algebra $\mathfrak{M}_{\mathcal{F}}Y$, defined by setting $(\mathfrak{M}_{\mathcal{F}}Y)\{B\} := \mathrm{map}_{\mathcal{C}}(B, Y)$ for each $B \in \mathbf{\Theta}_{\mathcal{F}}$. When $\mathcal{F}$ is generated by A, we write $\mathfrak{M}_A Y$ for $\mathfrak{M}_{\mathcal{F}}Y$.

1.5. **Example.** When $\mathcal{C} = \mathcal{T}_*$, $A = \mathbf{S}^1$, and $\kappa = \omega$, we see that $\mathbf{\Theta}_A$ is the full sub-simplicial category of $\mathcal{T}_*$ whose objects are finite wedges of spheres. In this case an A-mapping algebra is an enriched version of a Π-algebra (cf. [Sto, §4]) and a *realizable A-mapping algebra,* denoted by $\mathfrak{M}_A Y$, is just a loop space $Z = \Omega Y$ equipped with an action of all loop spaces of wedges of spheres on products of iterated loops of Z.

1.6. **Definition.** If A is a fixed cofibrant object A in a pointed simplicial model category $\mathcal{C}$ as above, we say that a map $f : X \to Y$ in $\mathcal{C}$ is an *A-equivalence* if the induced map $f_* : \mathrm{map}_{\mathcal{C}}(A, X) \to \mathrm{map}_{\mathcal{C}}(A, Y)$ is a weak equivalence of pointed simplicial sets. In particular, the natural map $\mathrm{CW}_A Y \to Y$ (cf. [DF, §2 A] is an A-equivalence.

Such an A-equivalence induces a weak equivalence of A-mapping algebras $f_* : \mathfrak{M}_A X \to \mathfrak{M}_A Y$, in the obvious sense.

1.7. **Definition.** A *free $\mathcal{F}$-mapping algebra* is one of the form $\mathfrak{M}_{\mathcal{F}}B$ for $B \in \mathbf{\Theta}_{\mathcal{F}}$.

1.8. **Lemma** (cf. [BB1, 8.13]). *If $\mathfrak{Y}$ is an $\mathcal{F}$-mapping algebra and $\mathfrak{M}_A B$ is a free $\mathcal{F}$-mapping algebra (for $B \in \mathbf{\Theta}_{\mathcal{F}}$), there is a natural isomorphism*

$$\Phi : \mathrm{map}_{\mathcal{M}ap_{\mathcal{F}}}(\mathfrak{M}_{\mathcal{F}}B, \mathfrak{Y}) \xrightarrow{\cong} \mathfrak{Y}\{B\} ,$$

with $\Phi(\mathfrak{f}) = \mathfrak{f}(\mathrm{Id}_B) \in \mathfrak{Y}\{B\}_0$ for any $\mathfrak{f} \in \mathrm{Hom}_{\mathcal{M}ap_{\mathcal{F}}}(\mathfrak{M}_{\mathcal{F}}B, \mathfrak{Y}) = \mathrm{map}_{\mathcal{M}ap_{\mathcal{F}}}(\mathfrak{M}_{\mathcal{F}}B, \mathfrak{Y})_0$.

Proof. This follows from the strong Yoneda Lemma for enriched categories (see [K, 2.4]). □

Evidently, the values of any A-mapping algebra $\mathfrak{X}$ on the *objects* of $\mathbf{\Theta}_A$ is completely determined by the single simplicial set $Y := \mathfrak{X}\{A\}$, since all other values are obtained from Y by (1.2). This leads to the following

1.9. **Definition.** A *discrete A-mapping algebra* is a function $\mathfrak{X} : \mathrm{Obj}(\mathbf{\Theta}_A) \to \mathcal{S}_*^{\mathrm{Kan}}$, written as before $B \mapsto \mathfrak{X}\{B\}$, equipped with isomorphisms

(1.10) $$\theta_A : \mathfrak{X}\{\Sigma A\} \to \Omega\mathfrak{X}\{A\}$$

as in (1.2). The category of such is denoted by $\mathcal{M}ap_A^{\delta}$, and there is a forgetful functor $U : \mathcal{M}ap_A \to \mathcal{M}ap_A^{\delta}$, written $\mathfrak{X} \mapsto \mathfrak{X}^{\delta}$.

1.11. Extended A-mapping algebras. We can of course further extend the domain of definition of a discrete A-mapping algebra to the collection $\mathcal{C}_A$ of all objects of $\mathcal{C}$ obtained recursively from A by pointed (homotopy) colimits. However, lifting such an "extended discrete A-mapping algebra " to a simplicial functor on the full simplicial subcategory of $\mathcal{C}$ with objects $\mathcal{C}_A$ requires a great deal of additional structure, so we want to be able to restrict the type of colimits used to the minimum actually required for our purposes. In particular, we need to restrict to a *set* of objects in $\mathcal{C}_A$.

In this connection we recall the following:

1.12. Definition. An object A in a pointed simplicial model category $\mathcal{C}$ is called λ-*small*, for an ordinal λ, if the canonical map

$$(1.13) \qquad \operatorname{colim}_\alpha \ \operatorname{map}_{\mathcal{C}}(B, X_\alpha) \ \to \ \operatorname{map}_{\mathcal{C}}(B, \operatorname{colim}_\alpha X_\alpha)$$

is an isomorphism for any sequence $(X_\alpha)_{\alpha < \lambda}$ in $\mathcal{C}$ and any object B of the form $B = \Sigma^i A$ $(i \geq 0)$.

This always holds for any A (for some limit ordinal λ) when $\mathcal{C}$ is $\mathcal{S}_*$ or $\mathcal{T}_*$ (cf. [DF, §1.B.3]); we shall always assume the source $A \in \mathcal{C}$ for any putative mapping space is λ-small for some λ; the least such infinite ordinal will be denoted by $\lambda(A)$. In particular, when $A \in \mathcal{S}$ is a finite CW complex, we have $\lambda(A) = \omega$.

1.14. Definition. Given $A \in \mathcal{C}$ and a set $\mathcal{I}$ of diagrams $I \to \mathcal{C}$, let $\Theta^{\mathcal{I}}_A$ denote the smallest sub-simplicial category of $\mathcal{C}$ containing A and closed under pointed colimits over the diagrams in $\mathcal{I}$.

We shall always assume that all objects in $\Theta^{\mathcal{I}}_A$ are obtained from A by iterated pointed colimits over diagrams from $\mathcal{I}$, and $\Theta^{\mathcal{I}}_A$ is closed under suspensions and coproducts of cardinality $< \kappa$, for some limit cardinal κ.

We define an *extended A-mapping algebra* $\mathfrak{X}$ with respect to $\mathcal{I}$ to be a simplicial functor $\mathfrak{X} : (\Theta^{\mathcal{I}}_A)^{\mathrm{op}} \to \mathcal{S}^{\mathrm{Kan}}_*$ such that:

$$(1.15) \qquad \mathfrak{X}\{B\} \ := \ \lim_{i \in I} \mathfrak{X}\{B_i\}$$

whenever $B = \operatorname{colim}_{i \in I} B_i$ for some diagram $I \to \mathcal{C}$ in $\mathcal{I}$. By assumption $\mathfrak{X}$ will in particular satisfy (1.2). The category of all such extended A-mapping algebras will be denoted by $\mathcal{M}ap^{\mathcal{I}}_A$.

1.16. Lemma. *If $\mathfrak{X}$ is an extended A-mapping algebra such that $\mathfrak{X}\{A\}$ is κ-small in $\mathcal{S}_*$, then $\mathfrak{X}$ is κ-small in $\mathcal{M}ap^{\mathcal{I}}_A$.*

Proof. A map of extended A-mapping algebras $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is determined by the map of pointed simplicial sets $\mathfrak{f}\{A\} : \mathfrak{X}\{A\} \to \mathfrak{Y}\{A\}$, because by (1.2) we have $\mathfrak{f}\{\coprod_i A_i\} = \prod_i \mathfrak{f}\{A_i\}$, and similarly for the other colimits in $\mathcal{I}$ by (1.15). $\qquad\square$

1.17. Definition. Given $A \in \mathcal{C}$ and $\mathcal{I}$ as in §1.14, a *discrete extended A-mapping algebra* (with respect to $\Theta^{\mathcal{I}}_A$) is a function $\mathfrak{Y} : \Theta^{\mathcal{I}}_A \to \mathcal{S}^{\mathrm{Kan}}_*$, written $B \mapsto \mathfrak{Y}\{B\}$, which takes the colimits in $\Theta^{\mathcal{I}}_A$ (of the diagrams in $\mathcal{I}$) to limits in $\mathcal{S}_*$ (compare §1.9).

Note that such an $\mathfrak{Y}$ is completely determined by the pointed simplicial set $X := \mathfrak{Y}\{A\}$ (which we assume to be fibrant), with $\mathfrak{Y}\{\operatorname{colim}_I B_i\} := \lim_I \mathfrak{Y}\{B_i\}$ for every diagram $I \to \mathcal{C}$ in $\mathcal{I}$. We therefore denote this discrete extended A-mapping algebra $\mathfrak{Y}$ by $\mathfrak{D}^e X$.

Letting $\mathcal{O}$ denote the set of objects of $\Theta_A^{\mathcal{I}}$, we define the $(\mathcal{S}_*, \mathcal{O})$-category $\mathcal{C}_X$ to be the sub-simplicial category of $\mathcal{S}_*^{\mathrm{op}}$ whose objects are in the image of $\mathfrak{D}^e X$ (cf. §0.3). Thus if $B = \mathrm{colim}_I B_i$ and $C = \mathrm{colim}_J C_j$ are two objects in $\mathcal{O}$, we have

$$
\begin{aligned}
\mathrm{map}_{\mathcal{C}_X}(B,C) &:= \mathrm{map}_{\mathcal{S}_*}(\lim_J \mathfrak{D}^e X\{C_j\},\ \lim_I \mathfrak{D}^e X\{B_i\}) \\
&= \lim_I \mathrm{map}_{\mathcal{S}_*}(\lim_J \mathfrak{D}^e X\{C_j\},\ \mathfrak{D}^e X\{B_i\}) .
\end{aligned}
$$
(1.18)

By analogy with [May, Lemma 1.4] we have:

1.19. **Lemma.** *For any fibrant $X \in \mathcal{S}_*$, there is a natural one-to-one correspondence between extended A-mapping algebras $\mathfrak{Y} \in \mathrm{Map}_A^{\mathcal{I}}$ with $\mathfrak{Y}\{A\} = X$ and $(\mathcal{S}_*, \mathcal{O})$-maps $f : \Theta_A^{\mathcal{I}} \to \mathcal{C}_X$.*

Proof. Since all objects in $\mathcal{O}$ are obtained from A by iterated colimits, we see from (1.18) that all the mapping spaces in $\mathcal{C}_X$ are obtained from those of the form

$$
\mathrm{map}_{\mathcal{S}_*}(\lim_J \mathfrak{D}^e X\{C_j\}, X)
$$
(1.20)

by iterated limits. The action of $\mathrm{map}_{\mathcal{C}}(A, \mathrm{colim}_J C_j)$ on $\mathfrak{Y}\{\mathrm{colim}_J C_j\}$ is then given by

$$
\begin{array}{ccccc}
\mathrm{map}_{\mathcal{C}}(A, \mathrm{colim}_J C_j) & \times & \mathfrak{Y}\{\mathrm{colim}_J C_j\} & \longrightarrow & \mathfrak{Y}\{A\} \\
\downarrow{\scriptstyle f} & & \| & & \| \\
\mathrm{map}_{\mathcal{S}_*}(\lim_J \mathfrak{D}^e X\{C_j\}, X) & \times & \lim_J \mathfrak{D}^e X\{C_j\} & \xrightarrow{\ \circ\ } & X
\end{array}
$$

where the bottom map is composition in $\mathcal{C}_X$. $\qquad\square$

2. The Stover construction

In [Sto, §2], Stover showed how one could use a simple construction in order to encode the homotopy type of a (pointed connected) topological space $Y \in \mathcal{T}_*$ by means of wedges of spheres: this was later used in [DKSt1] to define cofibrant replacements in the E^2-model category of simplicial sets.

Stover's basic idea was to start with a wedge of n-spheres $\mathbf{S}_{(f)}^n$ of $(n \geq 1)$, one for each map $f : \mathbf{S}^n \to Y$. If f is null-homotopic, attach a cone $C\mathbf{S}^n$ to $\mathbf{S}_{(f)}^n$ for each null-homotopy $F : f \sim *$ of f. The resulting space LY is homotopy equivalent to a wedge of spheres (of various dimensions), and is equipped with the obvious "tautological" map $\varepsilon : LY \to Y$. Iterating this yields a simplicial resolution of Y: that is, a simplicial space $V_\bullet$ whose realization $\|V_\bullet\|$ is weakly equivalent to the original Y, with each V_n homotopy equivalent to a wedge of spheres. Moreover, the cardinality of this wedge (pointed coproduct) is bounded by the number of 0- and 1-simplices in the mapping spaces $\mathrm{map}_*(\mathbf{S}^n, Y)$.

It turns out that the functor L factors through the mapping algebra functor $\mathfrak{M}_A$ (for $A = \mathbf{S}^1$), and in fact $V_\bullet$, and thus $\|V_\bullet\|$, is determined up to homotopy by the mapping algebra $\mathfrak{M}_A Y$ (cf. [BB1, §9]) – that is, by the loop space ΩY, with the action of mapping spaces between spheres on it (cf. §1.5). This is not very surprising, since this action includes inter alia the A_∞ structure on ΩY, from which we can recover Y up to weak equivalence by the methods of [Sta1, May].

However, the Stover construction can be applied to spaces other than $\mathbf{S}^1$ and its suspensions (although it needs to be modified when we start with a non-co-H-space A, since in this case it is not enough to encode the null-homotopies – we need all (pointed) homotopies). Thus we may hope that the A-mapping algebra structure on $\mathrm{map}_*(A, Y)$, when described properly in terms of the actual spaces appearing in the Stover constructions (rather than the strict pointed coproducts and suspensions of Definition 1.1), will allow us to recover $\mathrm{CW}_A Y$ from it.

This motivates the following series of definitions:

2.1. **Definition.** Let A be a cofibrant object in $\mathcal{C}$, as in §0.4, and κ a cardinal. An *elementary A-Stover object* (for κ) is any colimit $B = \mathrm{colim}_I X$, where I is an indexing category consisting of a disjoint union of:

   (a) A set of objects S of cardinality $< \kappa$;
   (b) For each pair $(u, v) \in S \times S$, a (possibly empty) set of objects $T_{(u,v)}$ of cardinality $< \kappa$;

   For each object $t \in T_{(u,v)}$, I has two morphisms $i_u : u \to t$ and $i_v : v \to t$.
   The diagram X has $X(u) \cong A$ for each $u \in S$, $X(t) \cong A \otimes \Delta[1]$ for each $t \in T_{(u,v)}$, where $X(i_u)$ is identified with $i_0 : A \otimes \{0\} \hookrightarrow A \otimes \Delta[1]$ and $X(i_v)$ identified with $i_1 : A \otimes \{1\} \hookrightarrow A \otimes \Delta[1]$ for any $(u, v) \in S \times S$.
   An *A-Stover object* (for κ) is a coproduct in $\mathcal{C}$ of a set of cardinality $< \kappa$ of elementary $\Sigma^i A$-Stover objects for κ, for various objects $i \geq 0$.

2.2. *Remark.* Note that if $\mathcal{C} = \mathcal{S}_*$ (or $\mathcal{T}_*$), the colimits described above are in fact (pointed) homotopy colimits. We shall assume this is true of the pointed model category $\mathcal{C}$ in which we are working; this will make the construction of Section 6 homotopy invariant.

Moreover, since $\mathcal{S}_*$ and $\mathcal{T}_*$ are pointed, the simplicial operation $X \otimes K$ is given by the half-smash of simplicial sets (or topological spaces):

$$(2.3) \qquad\qquad X \otimes K \;=\; X \rtimes K \;:=\; (X \times K)/(\ast \times K)$$

(which is itself pointed).

2.4. **The generalized Stover comonad.** Let $\mathcal{F} = \mathcal{F}^{\mathrm{St}}$ be the set of elementary Stover objects in $\mathcal{C}$ (for κ), and Θ_A^{St} the full sub-simplicial category of $\mathcal{C}^{\mathrm{op}}$ generated by $\mathcal{F}^{\mathrm{St}}$ under suspensions and coproducts of cardinality $< \kappa$. An $\mathcal{F}$-mapping algebra $\mathfrak{X}$ will be called a *Stover A-mapping algebra* (see §1.1), the category Stover A-mapping algebras will be denoted by $\mathcal{M}ap_A^{\mathrm{St}}$, and the discrete version (§1.9) by $\mathcal{M}ap_A^{\mathrm{St},\delta}$. The corresponding mapping functors are $\mathfrak{M}_A^{\mathrm{St}} : \mathcal{C} \to \mathcal{M}ap_A^{\mathrm{St}}$ and $\mathfrak{M}_A^{\mathrm{St},\delta} : \mathcal{C} \to \mathcal{M}ap_A^{\mathrm{St},\delta}$, respectively.

This allows us to define the *generalized Stover comonad* $L_A : \mathcal{C} \to \mathcal{C}$ by:

$$(2.5) \qquad\qquad L_A Y \;:=\; \coprod_{i \geq 0} \; \coprod_{\phi \in [\Sigma^i A, Y]} E_\phi \;,$$

for any $Y \in \mathcal{C}$, where for each $B := \Sigma^i A$ ($i \geq 0$), and homotopy class $\phi \in [B, Y] = \pi_0 \mathrm{map}_*(B, Y)$ in $\mathrm{ho}\,\mathcal{C}$, the elementary B-Stover object $E_\phi = \mathrm{colim}_I X$ is defined as in §2.1 with:

   (a) The set S consisting of all $f \in \mathrm{map}_{\mathcal{C}}(B, Y)_0$ with $[f] = \phi$;

(b) For each pair $(f, g) \in S \times S$, the set $T_{(f,g)}$ consisting of all homotopies $F \in \operatorname{map}_{\mathcal{C}}(B, Y)_1 = \operatorname{Hom}(B \rtimes \Delta[1], Y)$ with $d_0 F = f$ and $d_1 F = g$.

2.6. *Remark.* When A is a homotopy cogroup object (e.g., suspensions), we can make do with the original Stover construction, as defined in [Sto, §2], where for any A-mapping algebra $\mathfrak{Y}$:

$$(2.7) \quad L\mathfrak{Y} := \coprod_{i \geq 0} \coprod_{\phi \in \mathfrak{Y}\{\sigma^i A\}_0} \operatorname{colim}\left(\sigma^i A_{(\phi)} \xrightarrow{\text{inc}} (C\sigma^i A_{(\Phi)})_{\Phi \in \mathfrak{Y}\{\sigma^i A\}_1, d_0 \Phi = \phi, d_1 \phi = 0}\right),$$

that is, we use only cones indexed by null-homotopies, instead cylinders indexed by arbitrary homotopies (see also [BB1, §9]). We identify the copy of $\sigma^i A$ indexed by the 0 map itself with $*$, so that the colimit of cones attached to it becomes a coproduct of copies of $\sigma^{i+1} A$, which we identify with the appropriate summands associated to $\mathfrak{Y}\{\sigma^i A\}_0$.

Note, however, that this will not work without the cogroup condition, so that for arbitrary A we need the generalized version defined above.

2.8. **Definition.** Let J be the category having two objects j, j' and two non-identity maps $\phi, \psi : j \to j'$, and define a truncation functor $\rho : \mathcal{S} \to \mathcal{S}et^J$ by $\rho K := \{d_0, d_1 : K_1 \to K_0\}$. A *truncated discrete Stover A-mapping algebra* is a function $\mathfrak{X} : \operatorname{Obj}(\Theta_A^{\text{St}}) \to \mathcal{S}et^J$, equipped natural embeddings $\mathfrak{X}\{\Sigma A\}_0 \to \mathfrak{X}\{A\}_1$ as in (1.10).

2.9. **Lemma.** *The functor $L_A : \mathcal{C} \to \mathcal{C}$ is a comonad.*

Proof. Note that L_A factors as the composite $\tilde{L}_A \circ \rho \circ \mathfrak{M}_A^{\text{St},\delta}$, since from (2.5) *ff.* we see that $L_A Y$ depends only on the 1-skeleton of the discrete Stover A-mapping algebra $\mathfrak{M}_A^{\text{St},\delta} Y$. It is readily verified that $\tilde{L}_A$ is left adjoint to the truncated discrete Stover A-mapping algebra functor $\rho \mathfrak{M}_A^{\text{St},\delta}$ (compare [BB1, Proposition 9.12]). $\square$

2.10. *Remark.* The counit ε for this comonad is the tautological map $\varepsilon_Y : L_A Y \to Y$, which sends the copy of B indexed by ϕ to Y via the map ϕ, and the copy of $B \rtimes \Delta[1]$ indexed by Φ to Y via Φ (in the notation of §2.4). The comultiplication $\mu : L_A Y \to L_A L_A Y$ may also be described explicitly.

2.11. **The hierarchy of mapping algebras.** So far we have described a hierarchy of notions of mapping algebras, each needed at a different stage in our reconstruction process:

(a) The truncated discrete Stover A-mapping algebra $\rho \mathfrak{M}_A^{\text{St},\delta} Y$ contain the minimal amount of information needed to define the Stover construction $L_A Y$, and thus "resolve" Y by wedges of half-suspensions of A.

(b) The various discrete A-mapping algebras, such as $\mathfrak{M}_A^{\text{St},\delta} Y$, can be recovered from $X := \operatorname{map}_*(A, Y)$ alone by means of appropriate (homotopy) limits.

(c) Ordinary A-mapping algebras such as $\mathfrak{M}_A Y$ (with variants such as the Stover A-mapping algebra $\mathfrak{M}_A^{\text{St}} Y$) contain a great deal of additional information about $X := \operatorname{map}_*(A, Y)$. However, these suffice to reconstruct Y only for very special choices of A, such as wedges of spheres in $\mathcal{T}_*$ (see [Ba, BD, BB1] and §6.21 below).

(d) In general, in order to recover Y (up to A-equivalence) we will have to allow additional (homotopy) colimits in define suitable *extended* A-mapping algebras, with respect to an extended theory $\Theta_A^{\mathcal{I}}$, as in §1.11. However, we have to be careful in making these choices, so as to guarantee that $\Theta_A^{\mathcal{I}}$ is small as an enriched category (as well as being as economical as possible in terms of the extra structure imposed on X), on the one hand – but at the same time sufficiently large to permit recovering Y. The necessary choices are described in §4.1 below.

3. Monads and algebras

Given a monad $\Theta : \mathcal{C} \to \mathcal{C}$ over any category $\mathcal{C}$, we can define algebras over Θ as in [McL, VI, §2], as an alternative to "universal algebras", described in terms of operations and relations. Unfortunately, the functor $\mathfrak{T}_A^{\mathrm{St},\delta} := \mathfrak{M}_A^{\mathrm{St},\delta} \circ \tilde{L}_A \circ \rho : \mathcal{M}ap_A^{\mathrm{St},\delta} \to \mathcal{M}ap_A^{\mathrm{St},\delta}$ is not a monad on discrete Stover A-mapping algebras. However, it still fits into the following framework:

3.1. **Definition.** Let $\Theta_{\mathcal{F}}$ be some sub-simplicial category of $\mathcal{C}$ as in §1.1 (possibly discrete), such that $\mathcal{F}^{\mathrm{St}} \subseteq \mathcal{F}$, and let $\widehat{\rho} : \mathcal{M}ap_{\mathcal{F}} \to \rho \mathcal{M}ap_A^{\mathrm{St},\delta}$ be the induced forgetful functor from $\mathcal{F}$-mapping algebras to truncated discrete Stover A-mapping algebras (for some fixed κ). Let $\mathfrak{T}_{\mathcal{F}}$ denote the endofunctor $\mathfrak{M}_{\mathcal{F}} \circ \tilde{L}_A \circ \widehat{\rho} : \mathcal{M}ap_{\mathcal{F}} \to \mathcal{M}ap_{\mathcal{F}}$. A $\mathfrak{T}_{\mathcal{F}}$-*algebra* is an $\mathcal{F}$-mapping algebra $\mathfrak{X}$ equipped with a map of $\mathcal{F}$-mapping algebras $h_{\mathfrak{X}} : \mathfrak{T}_{\mathcal{F}}\mathfrak{X} \to \mathfrak{X}$ making the diagram

(3.2)
$$
\begin{CD}
\mathfrak{T}_{\mathcal{F}}\mathfrak{T}_{\mathcal{F}}\mathfrak{X} @>{\mathfrak{M}_{\mathcal{F}}\varepsilon_{\tilde{L}_A\widehat{\rho}\mathfrak{X}}}>> \mathfrak{T}_{\mathcal{F}}\mathfrak{X} \\
@V{\mathfrak{T}_{\mathcal{F}}(h_{\mathfrak{X}})}VV @VV{h_{\mathfrak{X}}}V \\
\mathfrak{T}_{\mathcal{F}}\mathfrak{X} @>>{h_{\mathfrak{X}}}> \mathfrak{X}
\end{CD}
$$

commute, with:

(3.3)
$$
\widehat{\rho}h_{\mathfrak{X}} \circ \eta_{\mathfrak{X}} = \mathrm{Id}_{\widehat{\rho}\mathfrak{X}} ,
$$

where

(3.4)
$$
\eta_{\mathfrak{X}} : \widehat{\rho}\mathfrak{X} \longrightarrow \widehat{\rho}\mathfrak{T}_{\mathcal{F}}\mathfrak{X} = \widehat{\rho}\mathfrak{M}_{\mathcal{F}}\tilde{L}_A\widehat{\rho}\mathfrak{X} = \rho\mathfrak{M}_A^{\mathrm{St},\delta}\tilde{L}_A\widehat{\rho}\mathfrak{X}
$$

is the unit of the monad $\rho \circ \mathfrak{M}_A^{\mathrm{St},\delta} \circ \tilde{L}_A : \rho\mathcal{M}ap_A^{\mathrm{St},\delta} \to \rho\mathcal{M}ap_A^{\mathrm{St},\delta}$ on truncated discrete Stover A-mapping algebras.

3.5. *Remark.* Note that if $\mathfrak{X} = \mathfrak{M}_{\mathcal{F}}Y$ for some $Y \in \mathcal{C}$, then:

(3.6)
$$
\begin{CD}
L_A L_A Y @>{\varepsilon_{\tilde{L}_A\widehat{\rho}\mathfrak{X}}}>> L_A Y \\
@V{L_A(\varepsilon_{\mathfrak{X}})}VV @VV{\varepsilon_Y}V \\
L_A Y @>>{\varepsilon_Y}> Y
\end{CD}
$$

commutes by Lemma 2.9 and [McL, VI, §1]. Thus

$$
h_{\mathfrak{X}} := \mathfrak{M}_{\mathcal{F}}(\varepsilon_Y) : \mathfrak{M}_{\mathcal{F}}L_A Y \to \mathfrak{M}_{\mathcal{F}}Y
$$

indeed makes (3.2) commute, by applying $\mathfrak{M}_{\mathcal{F}}$ to (3.6); and (3.3) also holds (cf. [McL, VI, §2]).

3.7. **Proposition.** *For* $\mathcal{F}^{\mathrm{St}} \subseteq \mathcal{F}$ *as above, every* $\mathcal{F}$-*mapping algebra* $\mathfrak{X}$ *has a natural* $\mathfrak{T}_{\mathcal{F}}$-*algebra structure map of* $\mathcal{F}$-*mapping algebras* $h_{\mathfrak{X}} : \mathfrak{T}_{\mathcal{F}}\mathfrak{X} \to \mathfrak{X}$.

Proof. First, we show that there is a natural map $h = h_{\mathfrak{X}} : \mathfrak{T}_{\mathcal{F}}\mathfrak{X} \to \mathfrak{X}$, determined by the pre-compositions with maps in $\boldsymbol{\Theta}_A^{\mathrm{St}}$: since $B := \tilde{L}_A\rho\mathfrak{X}$ is in $\boldsymbol{\Theta}_A^{\mathrm{St}} \subseteq \boldsymbol{\Theta}_{\mathcal{F}}$, the $\mathcal{F}$-mapping algebra $\mathfrak{M}_{\mathcal{F}}\tilde{L}_A\widehat{\rho}\mathfrak{X}$ overlying $\mathfrak{T}_{\mathcal{F}}\mathfrak{X}$ is free, so by Lemma 1.8, in order to define h as a map of $\mathcal{F}$-mapping algebras we just need to choose an element in $\mathfrak{X}\{B\}_0$. Since B is a coproduct, by (2.5), we need elements h_ϕ in $\widehat{\rho}\mathfrak{X}\{E_\phi\}_0$ for each $\phi \in \pi_0\widehat{\rho}\mathfrak{X}\{A\}$, and since E_ϕ is in turn a colimit, h_ϕ is determined by the tautological elements $f \in \widehat{\rho}\mathfrak{X}\{A'\}_0 = \mathfrak{X}\{A\}_0$ for each f representing ϕ.

To check that (3.2) commutes, note that $C := \tilde{L}_A\widehat{\rho}\mathfrak{T}_{\mathcal{F}}\mathfrak{X}$ is again in $\boldsymbol{\Theta}_A^{\mathrm{St}} \subseteq \boldsymbol{\Theta}_{\mathcal{F}}$, so the $\mathcal{F}$-mapping algebra $\mathfrak{T}_{\mathcal{F}}\mathfrak{T}_{\mathcal{F}}\mathfrak{X}$ is again free, and we need only check that $\mathfrak{T}_{\mathcal{F}}\mathfrak{T}_{\mathcal{F}}\mathfrak{X}\{C\}$ is sent to the same place in $\mathfrak{X}\{C\}$. Again, C is a coproduct of objects of the form E_ϕ, which are in turn colimits of objects B or $B \rtimes \Delta[1]$ with $B = \Sigma^k A$. Each such B is indexed by a map $f : B \to \bigvee_i E_{\phi_i} \subseteq \tilde{L}_A\widehat{\rho}\mathfrak{X}$ (or $F : B \rtimes \Delta[1] \to \bigvee_i E_{\phi_i}$), with E_{ϕ_i} itself a colimit of objects $B_{ik} = \Sigma^{n_{ik}}A$ (or $B_{ik} \rtimes \Delta[1]$) indexed by "formal maps" $f_{ik} \in \mathfrak{X}\{B_{ji}\}$ (or $F_{ik} \in \mathfrak{X}\{B_{ik} \rtimes \Delta[1]\}$). Together the maps f_{ik} and F_{ik} (for all j) induce a "formal map" $g_i \in \mathfrak{X}\{E_{\phi_i}\}$ (for each i), and thus a single element $g \in \mathfrak{X}\{\bigvee_i E_{\phi_i}\}$.

Now if $j : B \to E_\phi \hookrightarrow C$ is the structure map for the colimit, followed by the inclusion, thought of as a 0-simplex in $(\mathfrak{T}_{\mathcal{F}}\mathfrak{T}_{\mathcal{F}}\mathfrak{X})\{B\}$, then $\mathfrak{M}_{\mathcal{F}}\varepsilon_{\tilde{L}_A\widehat{\rho}\mathfrak{X}}$ (the top horizontal map in (3.2)) takes j to the map $f : B \to \bigvee_i E_{\phi_i}$, thought of as a 0-simplex in $(\mathfrak{M}_{\mathcal{F}}\tilde{L}_A\widehat{\rho}\mathfrak{X}) \in B$. Then $h_{\mathfrak{X}}$ (the right vertical map in (3.2)) takes f to $g \circ f \in \mathfrak{X}\{B\}$, since h is a map of Stover A-mapping algebras by construction.

On the other hand, $\mathfrak{T}_{\mathcal{F}}(h_{\mathfrak{X}})$ (the left vertical map in (3.2)) takes j to the map ι_B identifying B with the copy of B in the colimit $\tilde{L}_A\widehat{\rho}\mathfrak{X}$ indexed by the map $g \circ f \in \mathfrak{X}\{B\}$ (by definition of the functor $\mathfrak{M}_{\mathcal{F}}$, and thus of $\mathfrak{T}_{\mathcal{F}}$), and then $h_{\mathfrak{X}}$ (the bottom horizontal map in (3.2)) takes ι_B to $g \circ f \in \mathfrak{X}\{B\}$ by the definition above. Thus we see that (3.2) commutes on generators, and since all maps are maps of $\mathcal{F}$-mapping algebras, the diagram commutes.

To verify (3.3),] by adjunction it suffices to check that

$$\tilde{L}_A\widehat{\rho}h_{\mathfrak{X}} \circ \tilde{L}_A\eta_{\mathfrak{X}} = \mathrm{Id}_{\tilde{L}_A\widehat{\rho}\mathfrak{X}} ;$$

indeed, given $f \in \widehat{\rho}\mathfrak{X}\{B\}_0$, $\tilde{L}_A\eta_{\mathfrak{X}}$ takes the copy B_f of B in $\tilde{L}_A\widehat{\rho}\mathfrak{X}$ isomorphically to the copy of B_ι in $\tilde{L}_A\rho\mathfrak{M}_A^{\mathrm{St},\delta}\tilde{L}_A\widehat{\rho}\mathfrak{X}$ indexed by the inclusion $\iota : B_f \hookrightarrow \tilde{L}_A\widehat{\rho}\mathfrak{X}$, which $\tilde{L}_A\widehat{\rho}h_{\mathfrak{X}}$ maps back to B_f by construction. Similarly for $F \in \widehat{\rho}\mathfrak{X}\{B\}_1$. $\square$

3.8. *Remark.* The converse of Proposition 3.7 is also true, but as we shall not need this fact, we omit the proof.

4. THE INDUCTIVE COLIMIT CONSTRUCTION

We now describe a transfinite inductive process for recovering Y (up to A-equivalence) from the mapping space $X = \mathrm{map}(A, Y)$, equipped with the extra structure defined in the previous sections.

4.1. **Induction assumptions.** For any ordinal $\alpha < \lambda$, at the α-th stage of the induction we assume that we have constructed the previous approximations $Y^{(\beta)} \in \mathcal{C}$ for $\beta < \alpha$ and chosen a cardinal κ_α, as well as a set $\mathcal{F}^{(\alpha)}$ of objects in $\mathcal{C}$, containing $\bigcup_{\beta<\alpha} \mathcal{F}^{(\beta)}$, all A-Stover objects for κ_α (§2.1), and closed under:

(a) Suspensions and half-suspensions (cf. (2.3)):

(b) Coproducts of cardinality $\leq \kappa_\alpha$;

(c) Taking D' in the functorial factorization $C \to D' \to D$ of any map $C \to D$ in $\mathfrak{f}^{(\alpha)}$ as a cofibration followed by an acyclic fibration.

(d) Pushouts of diagrams $Z \leftarrow X \xrightarrow{i} Y$ when i is a cofibration between A-Stover objects for κ_β and Z is in $\mathcal{F}^{(\beta)}$, for $\beta < \alpha$.

(e) Colimits of sequences of cofibrations in $\bigcup_{\beta<\alpha} \mathcal{F}^{(\beta)}$ of length $< \alpha$.

Note that all the limits in question are homotopy colimits. and all objects are cofibrant (see Remark 2.2).

4.2. **Definition.** For any ordinal $\alpha \leq \lambda$, we let $\mathcal{C}_A^\alpha$ denote the sub-simplicial category of $\mathcal{C}$ with object set $\mathcal{F}^{(\alpha)}$, $\mathcal{M}ap^\alpha$ the corresponding category of extended A-mapping algebras (cf. §1.14), and $\mathfrak{M}_A^\alpha : \mathcal{C} \to \mathcal{M}ap^\alpha$ the mapping algebra functor.

4.3. *Remark.* The cardinal κ_α is determined inductively by letting $\kappa_0 := |X|^+$ be the least cardinal greater than the cardinality of the truncated simplicial set ρX, and thus a bound on the coproducts appearing in the A-Stover construction $\tilde{L}_A \rho \mathfrak{X}$ (in the notation of the proof of Lemma 2.9). In general, $\kappa_{\alpha+1}$ is the least upper bound on the cardinality of $\rho \mathfrak{M}_A Y$ for all $Y \in \mathcal{F}^{(\alpha)}$. For a limit ordinal α we set $\kappa_\alpha := \sup_{\beta<\alpha} \kappa_\beta$. We assume for simplicity that $\lambda \leq \kappa_0$.

4.4. *Remark.* Of course, any *realizable* A-mapping algebra $\mathfrak{M}_A Z$ extends uniquely to an $\mathcal{F}^{(\beta)}$-mapping algebra $\mathfrak{M}_A^\beta Z$ for any $\beta \leq \lambda$; in particular, we write $\widetilde{\mathfrak{T}}^\beta$ for $\mathfrak{M}_A^\beta \circ \tilde{L}_A : \rho \mathcal{M}ap_A \to \mathcal{M}ap^\beta$ (no longer a monad).

The main additional hypothesis needed to realize the original $\mathfrak{X}$ is that it can be extended to an $\mathcal{F}^{(\beta)}$-mapping algebra $\tilde{\mathfrak{X}}$ for each $\beta \leq \alpha$. Of course, this is trivially true for the *discrete* A-mapping algebra $\mathfrak{X}$, and thus the $\mathfrak{T}_A$-algebra structure map $h_{\mathfrak{X}} : \mathfrak{T}_A \mathfrak{X} \to \mathfrak{X}$ extends canonically to $\tilde{h}_{\mathfrak{X}} : \widetilde{\mathfrak{T}}^\beta \mathfrak{X} \to \tilde{\mathfrak{X}}$.

We can then make sense of the requirement that we have a map of $\mathcal{F}^{(\beta)}$-mapping algebras $\mathfrak{f}^{(\beta)} : \widetilde{\mathfrak{T}}^\beta Y^{(\beta)} \to \tilde{\mathfrak{X}}$ and a map $p^{(\beta)} : \tilde{L}_A \rho \mathfrak{X} \to Y^{(\beta)}$ in $\mathcal{C}$ for each $\beta \leq \alpha$, as well as a cofibration $i^{(\beta)} : Y^{(\beta)} \to Y^{(\beta+1)}$ in $\mathcal{C}$ such that

$$(4.5) \qquad \mathfrak{f}^{(\beta+1)} \circ \mathfrak{M}_A^{\beta+1} i^{(\beta)} = \mathfrak{f}^{(\beta)} \quad \text{and} \quad \mathfrak{f}^{(\beta+1)} \circ \mathfrak{M}_A^{\beta+1} p^{(\beta+1)} = \tilde{h}_{\mathfrak{X}}$$

for each $\beta < \alpha$, and

$$(4.6) \qquad \mathfrak{f}^{(\beta)} \circ \mathfrak{M}_A^\beta \varepsilon_{Y^{(\beta)}} = \tilde{h}_{\mathfrak{X}} \circ \widetilde{\mathfrak{T}}^\beta \mathfrak{f}^{(\beta)}$$

for each $\beta \leq \alpha$.

4.7. **The inductive construction.** We start with $Y^{(0)} := * \in \mathcal{C}$ as the 0-th approximation to Y, with $\mathcal{F}^{(0)} = \{A\}$. Assume by induction that we have obtained the above structure for some $\alpha < \lambda$. Now factor $\tilde{L}_A \rho \mathfrak{f}^{(\alpha)} : \tilde{L}_A \rho \mathfrak{M}_A Y^{(\alpha)} \to \tilde{L}_A \rho \mathfrak{X}$ (functorially) as

$$(4.8) \qquad \tilde{L}_A \rho \mathfrak{M}_A Y^{(\alpha)} \xhookrightarrow{\widetilde{\mathfrak{f}^{(\alpha)}}} \widetilde{L_A \mathfrak{X}} \xtwoheadrightarrow[\simeq]{\ell^{(\alpha)}} \tilde{L}_A \rho \mathfrak{X}$$

with $\widetilde{\mathfrak{f}^{(\alpha)}}$ a cofibration and $\ell^{(\alpha)}$ a trivial fibration. Because $\tilde{L}_A\rho\mathfrak{X}$ is cofibrant, $\ell^{(\alpha)}$ has a section $j^{(\alpha)} : \tilde{L}_A\rho\mathfrak{X} \to \widetilde{L_A\mathfrak{X}}$ (which is also a weak equivalence), with

$$(4.9) \qquad\qquad \ell^{(\alpha)} \circ j^{(\alpha)} = \mathrm{Id}_{\tilde{L}_A\rho\mathfrak{X}} .$$

We now define $Y^{(\alpha+1)}$, $p^{(\alpha+1)}$, and $i^{(\alpha)}$ by the pushout diagram:

$$(4.10)$$

$$
\begin{array}{ccccc}
L_A Y^{(\alpha)} & = & \tilde{L}_A\rho\mathfrak{M}_A Y^{(\alpha)} & \overset{\widetilde{\mathfrak{f}^{(\alpha)}}}{\longrightarrow} & \widetilde{L_A\mathfrak{X}} \\
{\scriptstyle \varepsilon_{Y^{(\alpha)}}}\downarrow & & & & \downarrow{\scriptstyle p^{(\alpha+1)}} \\
Y^{(\alpha)} & & \overset{i^{(\alpha)}}{\hookrightarrow} & \boxed{\mathrm{PO}} & \tilde{Y}^{(\alpha+1)}
\end{array}
$$

in $\mathcal{C}$ (which is thus a homotopy push out).

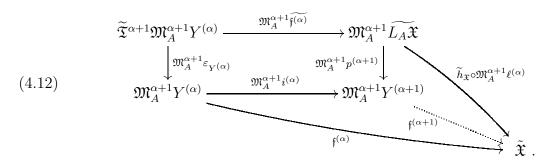Since all but the lower left corner of (4.10) are in $\mathcal{F}^{(\alpha)} \subseteq \mathcal{F}^{(\alpha+1)}$, the pushout $Y^{(\alpha+1)}$ is in $\mathcal{F}^{(\alpha+1)}$, by definition. Therefore, if we apply the $\mathcal{F}^{(\alpha+1)}$-mapping algebra-functor $\mathfrak{M}_A^{\alpha+1}$ to (4.10), we have a pushout diagram in $\mathcal{M}ap_{\mathcal{F}^{(\alpha+1)}}$, by Lemma 1.8.

Moreover, by (4.5) the following square commutes in $\mathcal{M}ap^\alpha$:

$$(4.11)$$

$$
\begin{array}{ccccccc}
& & \multicolumn{3}{c}{\overset{\widetilde{\mathfrak{T}^\alpha}\mathfrak{f}^{(\alpha)}}{\longrightarrow}} & & \\
\widetilde{\mathfrak{T}^\alpha}\mathfrak{M}_A^\alpha Y^{(\alpha)} & \overset{\mathfrak{M}_A^\alpha\widetilde{\mathfrak{f}^{(\alpha)}}}{\longrightarrow} & \mathfrak{M}_A^\alpha\widetilde{L_A\mathfrak{X}} & \overset{\mathfrak{M}_A^\alpha\ell^{(\alpha)}}{\longrightarrow} & \widetilde{\mathfrak{T}^\alpha}\mathfrak{X} & = & \mathfrak{L}_A\mathfrak{X} \\
{\scriptstyle \mathfrak{M}_A^\alpha\varepsilon_{Y^{(\alpha)}}}\downarrow & & & & & & \downarrow{\scriptstyle \tilde{h}_\mathfrak{X}} \\
\mathfrak{M}_A^\alpha Y^{(\alpha)} & & \overset{\mathfrak{f}^{(\alpha)}}{\longrightarrow} & & & & \mathfrak{X},
\end{array}
$$

and since all but the lower right corner are free $\mathcal{F}^{(\alpha)}$-mapping algebras, (4.11) extends automatically to a commuting square in $\mathcal{M}ap^{\alpha+1}$ (see §1.11).

Therefore, there is a dotted map of $\mathcal{F}^{(\alpha+1)}$-mapping algebras making the following diagram commute:

$$(4.12)$$

$$
\begin{array}{ccc}
\widetilde{\mathfrak{T}^{\alpha+1}}\mathfrak{M}_A^{\alpha+1} Y^{(\alpha)} & \overset{\mathfrak{M}_A^{\alpha+1}\widetilde{\mathfrak{f}^{(\alpha)}}}{\longrightarrow} & \mathfrak{M}_A^{\alpha+1}\widetilde{L_A\mathfrak{X}} \\
{\scriptstyle \mathfrak{M}_A^{\alpha+1}\varepsilon_{Y^{(\alpha)}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{M}_A^{\alpha+1}p^{(\alpha+1)}} \\
\mathfrak{M}_A^{\alpha+1} Y^{(\alpha)} & \overset{\mathfrak{M}_A^{\alpha+1}i^{(\alpha)}}{\longrightarrow} & \mathfrak{M}_A^{\alpha+1}Y^{(\alpha+1)} \\
\end{array}
$$

with $\tilde{h}_\mathfrak{X}\circ\mathfrak{M}_A^{\alpha+1}\ell^{(\alpha)}$, $\mathfrak{f}^{(\alpha+1)}$, and $\mathfrak{f}^{(\alpha)}$ mapping to $\tilde{\mathfrak{X}}$.

This completes the induction step, by defining $\mathfrak{f}^{(\alpha+1)}$.

4.13. Verifying the induction hypothesis. By construction (4.5) holds for $\alpha+1$; to see that (4.6) holds, too, consider the following diagram of $\mathcal{F}^{(\alpha+1)}$-mapping

algebras:

The top square commutes by applying $\widetilde{\mathfrak{T}}^{\alpha+1}$ to the right triangle of (4.12); the right square is (3.2) for $\widetilde{\mathfrak{T}}^{\alpha+1}$, which commutes by Proposition 3.7, since $\mathcal{F}^{\text{St}} \subseteq \mathcal{F}^{(\alpha+1)}$. The left and middle squares commute by naturality of ε; and the bottom square is the right triangle of (4.12). Therefore, the outer square commutes.

If α is a limit ordinal, we let $Y^{(\alpha)} := \text{colim}_{\beta < \alpha} Y^{(\beta)}$, with the structure maps $i^{(\beta)} : Y^{(\beta)} \to Y^{(\alpha)}$ for each $\beta < \alpha$. Note that by construction $\tilde{L}_A \rho \mathfrak{X} = \text{colim}_{\alpha < \beta} \tilde{L}_A \rho \mathfrak{X}$, since $\mathcal{F}^{(\alpha)} = \bigcup_{\alpha < \beta} \mathcal{F}^{(\beta)}$, so we may set $p^{(\alpha)} := \text{colim}_{\alpha < \beta} p^{(\beta)}$ and $\mathfrak{f}^{(\alpha)} := \text{colim}_{\alpha < \beta} \mathfrak{f}^{(\beta)}$.

5. Realizing mapping algebras

Evidently, if $X = \text{map}_*(A, Y) \in \mathcal{S}_*$ is a pointed mapping space out of A, we can use (1.2) to define the A-mapping algebra $\mathfrak{X} = \mathfrak{M}_A Y$ with $\mathfrak{X}\{A\} = X$. More generally, $\mathfrak{X}$ can be made into an extended $\mathcal{F}$-mapping algebra, where $\mathcal{F}$ is the closure of $\{A\}$ in $\mathcal{C}$ with respect to any set of homotopy colimits constructed from A.

We now show that the A-mapping algebra-structure on X, combined with the constructions of Section 4 indeed allows us to recover Y (up to A-equivalence):

5.1. **Theorem.** *Let A be a cofibrant λ-small object in a model category $\mathcal{C}$ as in §0.4, and X a pointed simplicial set. If X can be equipped with an extended $\mathcal{F}^{(\lambda)}$-mapping algebra structure $\mathfrak{X}$ for the family $\mathcal{F} = \mathcal{F}^{(\lambda)}$ defined in Section 4, then the map of $\mathcal{F}$-mapping algebras $\mathfrak{f} := \mathfrak{f}^{(\lambda)} : \mathfrak{M}_A^\lambda Y^{(\lambda)} \to \hat{\mathfrak{X}}$ constructed there induces a weak equivalence $X \simeq \text{map}_\mathcal{C}(A, Y)$.*

Proof. It suffices to show that $\mathfrak{f}_\# : \pi_0((\mathfrak{M}_A Y^{(\lambda)})\{\Sigma^i A\}) \to \pi_0(\hat{\mathfrak{X}}\{\Sigma^i A\})$ is bijective for any $i \geq 0$.

(a) We can use the unit $\eta_\mathfrak{X} : \rho\mathfrak{X} \to \rho\mathfrak{M}_A \tilde{L}_A \rho\mathfrak{X}$ of (3.4) to define:

$$s^{(\alpha+1)} := \rho\mathfrak{M}_A p^{(\alpha+1)} \circ \rho\mathfrak{M}_A p^{(\alpha+1)} j^{(\alpha+1)} \circ \eta_\mathfrak{X} : \rho\mathfrak{X} \to \rho\mathfrak{M}_A Y^{(\alpha+1)}$$

for any $\alpha < \lambda$.

Since $\mathcal{F}^{\mathrm{St}} \subseteq \mathcal{F}^{(\lambda)}$, $\hat{\mathfrak{X}}$ is an $\mathfrak{T}_{\mathcal{F}}$-algebra by Proposition 3.7. We therefore have a commuting diagram:

(5.2)

$$
\begin{array}{c}
\rho\mathfrak{X} = \widehat{\rho}\hat{\mathfrak{X}} \xrightarrow{\eta_{\mathfrak{X}}} \widehat{\rho}\mathfrak{M}_A^{\mathrm{St},\delta}\tilde{L}_A\rho\mathfrak{X} = \widehat{\rho}\mathfrak{M}_A^{\alpha+1}\tilde{L}_A\rho\mathfrak{X} \xrightarrow{\widehat{\rho}\mathfrak{M}_A^{\alpha+1}j^{(\alpha+1)}} \widehat{\rho}\mathfrak{M}_A^{\alpha+1}\widetilde{L_A\mathfrak{X}}
\end{array}
$$

in which the left triangle commutes by (3.3), the upper right triangle commutes by (4.9), and the lower right square commutates by (4.12). This shows that

(5.3)
$$
\widehat{\rho}\mathfrak{f}^{(\alpha+1)} \circ s^{(\alpha+1)} = \mathrm{Id}_{\rho\mathfrak{X}} \ .
$$

(b) Thus we have a diagram

(5.4)

$$
\begin{array}{c}
\widehat{\rho}\mathfrak{M}_A Y^{(0)} \xrightarrow{\widehat{\rho}\mathfrak{M}_A i^{(0)}} \widehat{\rho}\mathfrak{M}_A Y^{(1)} \xrightarrow{\widehat{\rho}\mathfrak{M}_A i^{(1)}} \cdots \widehat{\rho}\mathfrak{M}_A Y^{(\alpha)} \xrightarrow{\widehat{\rho}\mathfrak{M}_A i^{(\alpha)}} \widehat{\rho}\mathfrak{M}_A Y^{(\alpha+1)} \xrightarrow{\widehat{\rho}\mathfrak{M}_A i^{(\alpha+1)}} \cdots
\end{array}
$$

of truncated extended A-mapping algebras in which the squares and lower triangles commute. By Lemma 1.16 we then obtain a map

$$
s^B = s^{(\lambda)}\{B\} : \rho\mathfrak{X}\{B\} \to \rho\mathfrak{M}_A Y^{(\lambda)}\{B\}
$$

for each $B = \Sigma^i A \quad (0 \leq i < \infty)$ with $\rho\mathfrak{f}\{B\} \circ s^B = \mathrm{Id}_{\rho\mathfrak{X}\{B\}}$. This shows that $\mathfrak{f}_{\#}$ is surjective.
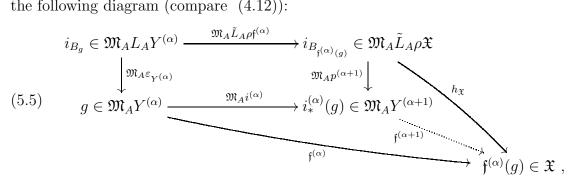
(c) Since $\rho\mathfrak{M}_A Y^{(\alpha)}\{B\}$ is just the 1-truncation of $\mathrm{map}(B, Y^{(\alpha)})$, by (1.13) we know

$$
\rho\mathfrak{M}_A Y^{(\lambda)}\{B\} = \mathrm{colim}_{\alpha<\lambda} \, \rho\mathfrak{M}_A Y^{(\alpha)}\{B\} \ ,
$$

and thus $\mathfrak{f}_{\#} = \mathrm{colim}_{\alpha<\lambda} \mathfrak{f}_{\#}^{(\alpha)} := \mathrm{colim}_{\alpha<\lambda} \pi_0\rho\mathfrak{f}^{(\alpha)}\{B\}$. Therefore, if $\mathfrak{f}_{\#}(\gamma) = \mathfrak{f}_{\#}(\gamma')$ for some $\gamma, \gamma' \in [B, Y^{(\lambda)}]$, then there is an $\alpha < \lambda$ and $\gamma_\alpha, \gamma'_\alpha \in [B, Y^{(\alpha)}]$, represented by $g, g' : B \to Y^{(\alpha)}$ in $\rho\mathfrak{M}_A Y^{(\alpha)}\{B\}_0$, with $\mathfrak{f}_{\#}^{(\alpha)}(\gamma_\alpha) = \mathfrak{f}_{\#}^{(\alpha)}(\gamma'_\alpha)$. In other words, there is a 1-simplex $\sigma \in \mathfrak{X}\{B\}_1$ with $d_0\sigma = \mathfrak{f}^{(\alpha)}(g)$ and $d_1\sigma = \mathfrak{f}^{(\alpha)}(g')$.

From the description in §3.5 we see that $\eta_{\rho\mathfrak{X}}(\sigma) = i_{B_\sigma} \in (\mathfrak{L}_A\mathfrak{X})\{B\}_1$, corresponding to the inclusion $B_\sigma \rtimes \Delta[1] \hookrightarrow \tilde{L}_A\mathfrak{X}$, with $d_0(i_{B_\sigma}) = i_{B_{\mathfrak{f}^{(\alpha)}(g)}} = \eta_{\rho\mathfrak{X}}(\mathfrak{f}^{(\alpha)}(g))$ and $d_1(i_{B_\sigma}) = i_{B_{\mathfrak{f}^{(\alpha)}(g')}} = \eta_{\rho\mathfrak{X}}(\mathfrak{f}^{(\alpha)}(g'))$. Similarly $\eta_{\rho\mathfrak{M}_A Y^{(\alpha)}}(g) = i_{B_g} \in (\mathfrak{L}_A\mathfrak{M}_A Y^{(\alpha)})\{B\}_0$ and $\eta_{\rho\mathfrak{M}_A Y^{(\alpha)}}(g') = i_{B_{g'}} \in (\mathfrak{L}_A\mathfrak{M}_A Y^{(\alpha)})\{B\}_0$, and all these 0-simplices match up in

the following diagram (compare (4.12)):

$$
\begin{array}{ccc}
i_{B_g} \in \mathfrak{M}_A L_A Y^{(\alpha)} & \xrightarrow{\;\mathfrak{M}_A \tilde{L}_A \rho \mathfrak{f}^{(\alpha)}\;} & i_{B_{\mathfrak{f}^{(\alpha)}(g)}} \in \mathfrak{M}_A \tilde{L}_A \rho \mathfrak{X} \\
\downarrow{\scriptstyle \mathfrak{M}_A \varepsilon_{Y^{(\alpha)}}} & & \downarrow{\scriptstyle \mathfrak{M}_A p^{(\alpha+1)}} \\
g \in \mathfrak{M}_A Y^{(\alpha)} & \xrightarrow{\;\mathfrak{M}_A i^{(\alpha)}\;} & i_*^{(\alpha)}(g) \in \mathfrak{M}_A Y^{(\alpha+1)}
\end{array}
$$

(5.5)

$$\mathfrak{f}^{(\alpha)}(g) \in \mathfrak{X} \;,$$

with $h_{\mathfrak{X}}$, $\mathfrak{f}^{(\alpha+1)}$, $\mathfrak{f}^{(\alpha)}$ maps into $\mathfrak{f}^{(\alpha)}(g) \in \mathfrak{X}$.

and similarly for g'. If we now set $\tau := \mathfrak{M}_A p^{(\alpha+1)}(i_{B_\sigma}) \in \mathfrak{M}_A Y^{(\alpha+1)}\{B\}_1$, then evidently $d_0\tau = i_*^{(\alpha)}(g)$ and $d_1\tau = i_*^{(\alpha)}(g')$, showing that $i_\#^{(\alpha)}(\gamma_\alpha) = i_\#^{(\alpha)}(\gamma'_\alpha)$, so a *fortiori* $\gamma = \gamma' \in [B, Y^{(\lambda)}]$.

Thus $\mathfrak{f}_\#$ is bijective, so $\mathfrak{f} : \mathfrak{M}_A Y^{(\lambda)} \to \mathfrak{X}$ is a weak equivalence of A-mapping algebras. We have thus shown in particular that the original space X is weakly equivalent to $\mathrm{map}_{\mathcal{C}}(A, Y^{(\lambda)})$. $\qquad\square$

Compare [DF, §2, Proposition B.1].

6. A homotopy invariant recognition principle

The recognition and recovery procedures described in Section 5 are rigid, inasmuch as they require that the structure on a given space X hold on the nose in order to deduce that $X \simeq \mathrm{map}_{\mathcal{C}}(A, Y)$ for some $Y \in \mathcal{C}$. We now describe a lax version of the latter, which is homotopy invariant in the sense that if the recognition principle holds for X, it also holds for any $X' \simeq X$.

6.1. Definition. For any ordinal $\alpha \leq \lambda$, let $\mathcal{C}_A^\alpha$ denote the simplicial subcategory of $\mathcal{C}$ with object set $\mathcal{O} := \mathcal{F}^{(\alpha)}$ defined in §4.1. A *lax (α-extended) theory* (for A) is any cofibrant $(\mathcal{S}, \mathcal{O})$-category $\widehat{\mathcal{C}}_A^\alpha$ weakly equivalent to $\mathcal{C}_A^\alpha$ (in the model category on $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$ defined in [DK, §1]).

A *lax (α-extended) A-mapping algebra* is a simplicial functor $\hat{\mathfrak{X}} : (\widehat{\mathcal{C}}_A^\alpha)^{\mathrm{op}} \to \mathcal{S}_*^{\mathrm{Kan}}$ such that on objects $\mathcal{O} = \mathcal{F}^{(\alpha)}$ $\hat{\mathfrak{X}}$ is isomorphic to the discrete extended A-mapping algebra $\mathfrak{D}^e X$ of §1.17, for $X := \hat{\mathfrak{X}}\{A\}$. In this case we also say that $\hat{\mathfrak{X}}$ constitutes a *lax (α-extended) A-mapping algebra structure* on X.

The category of lax α-extended A-mapping algebras is denoted by $\widehat{\mathcal{M}ap_A^\alpha}$.

6.2. Constructing lax extended A-theories. Lax α-extended theories for A can be constructed explicitly in several ways:

First, consider the category $\mathcal{F}_\bullet \mathcal{C}_A^\alpha$, enriched in bisimplicial sets, obtained by iterating the free category comonad $\mathcal{F} := F \circ U$ on each simplicial dimension of $\mathcal{C}_A^\alpha$, and let $\widehat{\mathcal{C}}_A^\alpha$ be the cofibrant $(\mathcal{S}, \mathcal{O})$-category obtained from $\mathcal{F}_\bullet \mathcal{C}_A^\alpha$ by taking the diagonal of each bisimplicial set. Note that $\widehat{\mathcal{C}}_A^\alpha$ is free in each simplicial dimension.

A minimal version $\widetilde{\mathcal{C}}_A^\alpha$ of this construction may be obtained as follows: start with $(\widetilde{\mathcal{C}}_A^\alpha)_0 := \mathcal{F}\pi_0 \mathcal{C}_A^\alpha$ which we can think of as a subcategory of $(\widehat{\mathcal{C}}_A^\alpha)_0$ by choosing a section $\sigma : \pi_0 \mathcal{C}_A^\alpha \to (\mathcal{C}_A^\alpha)_0$ (that is, a representative $\sigma[\phi] : B \to B'$ for each homotopy class $\phi \in [B, B']$ and $B, B' \in \mathcal{F}^{(\alpha)}$).

Next, choose a representative $\sigma[\Phi]$ for each homotopy class of homotopies between the chosen maps in $(\widetilde{\mathcal{C}}_A^\alpha)_0 \subseteq \widehat{\mathcal{C}}_A^\alpha)_0$ and composites thereof, and let $(\widetilde{\mathcal{C}}_A^\alpha)_1$ be the free category generated by these representatives $\sigma[\Phi]$ and $s_0(\widetilde{\mathcal{C}}_A^\alpha)_0$. Proceeding similarly for higher homotopies, we obtain a minimal model $\widetilde{\mathcal{C}}_A^\alpha$ for $\widehat{\mathcal{C}}_A^\alpha$.

We can describe $\widetilde{\mathcal{C}}_A^\alpha$ more formally by using a section $\sigma : \pi_0 \mathcal{C}_A^\alpha \to (\mathcal{C}_A^\alpha)_0$ to define $(\widetilde{\mathcal{C}}_A^\alpha)_0$ as above, but then noting that this determines

$$\operatorname{map}_{\widetilde{\mathcal{C}}_A^\alpha}(\Sigma B, B')_0 = \Omega \operatorname{map}_{\widetilde{\mathcal{C}}_A^\alpha}(B, B')_0 \subseteq \operatorname{map}_{\widetilde{\mathcal{C}}_A^\alpha}(B, B')_1$$

for any $B, B' \in \mathcal{F}^{(\alpha)}$, which can then be used to determine $\operatorname{map}_{\widetilde{\mathcal{C}}_A^\alpha}(B, B')_1$, and similarly for the higher simplices.

A third possibility is to replace $\widehat{\mathcal{C}}_A^\alpha$ by the free simplicial resolution $\mathcal{F}_\bullet(\mathcal{C}_A^\alpha)_0$ of the 0-simplices of $\mathcal{C}_A^\alpha$ – that is, of the ordinary (non-simplicial) category underlying $\mathcal{C}_A^\alpha$. This version certainly acts simplicially on any realizable mapping algebra, but depends on less data than the other two.

6.3. *Remark.* By definition, we have a weak equivalence of $(\mathcal{S}, \mathcal{O})$-categories $\phi : \widehat{\mathcal{C}}_A^\alpha \to \mathcal{C}_A^\alpha$, allowing us to pull back any strict extended A-mapping algebra $\mathfrak{X} : (\mathcal{C}_A^\alpha)^{\mathrm{op}} \to \mathcal{S}_*^{\mathrm{Kan}}$ to a lax one. If we use the first construction of §6.2 for $\widehat{\mathcal{C}}_A^\alpha$, the map ϕ is induced by the usual augmentation $\varepsilon : \mathcal{F}_\bullet \mathcal{C}_A^\lambda \to \mathcal{C}_A^\lambda$, which sends the free homotopies in $\mathcal{F}_\bullet(\mathcal{C}_A^\alpha)_0$ to identity homotopies.

In particular, for any $Y \in \mathcal{C}$ we have the *realizable* lax A-mapping algebra $\widehat{\mathfrak{M}}_A^\alpha Y$, as well as its discrete version $(\widehat{\mathfrak{M}}_A^\alpha Y)_\delta$.

A lax A-mapping algebra of the form $\widehat{\mathfrak{M}}_A^\alpha B$ for $B \in \mathcal{F}^{(\alpha)}$ is called *free* (cf. §1.7), and we have as in Lemma 1.8:

6.4. **Lemma.** *If $\hat{\mathfrak{Y}}$ is lax α-extended A-mapping algebra and $B \in \mathcal{F}^{(\alpha)}$, there is a natural isomorphism* $\operatorname{map}_{\widehat{Map}_A^\alpha}(\widehat{\mathfrak{M}}_A^\alpha B, \hat{\mathfrak{Y}}) \cong \hat{\mathfrak{Y}}\{B\}$.

6.5. **Lemma.** *Let B, C, and D be objects in $\mathcal{F}^{(\alpha)}$ for some $\alpha < \lambda$, $i : B \hookrightarrow C$ a cofibration and $f : B \to D$ any map in $\mathcal{C}$, with E the pushout of $D \leftarrow B \hookrightarrow C$ (which thus lies in $\mathcal{F}^{(\alpha+1)}$). Then*

(6.6)
$$
\begin{array}{ccc}
\widehat{\mathfrak{M}}_A^{\alpha+1} B & \xrightarrow{\ i_* \ } & \widehat{\mathfrak{M}}_A^{\alpha+1} C \\
\downarrow{\scriptstyle f_*} & & \downarrow \\
\widehat{\mathfrak{M}}_A^{\alpha+1} D & \longrightarrow & \widehat{\mathfrak{M}}_A^{\alpha+1} E
\end{array}
$$

is a homotopy pushout in $\widehat{Map}_A^{\alpha+1}$.

Proof. Note that since lax A-mapping algebras are modeled on $\widehat{\mathcal{C}}_A^\alpha$, rather than $\mathcal{C}_A^\alpha \subseteq \mathcal{C}$, the free lax A-mapping algebra functor $\widehat{\mathfrak{M}}_A^{\alpha+1}$ does not preserve colimits taken in $\mathcal{C}_A^\alpha$. However, since E is a *homotopy* pushout of $D \leftarrow B \hookrightarrow C$, by Lemma

1.8

(6.7)
$$
\begin{array}{ccc}
\mathfrak{M}_A^{\alpha+1} B & \xrightarrow{\;i_*\;} & \mathfrak{M}_A^{\alpha+1} C \\
\downarrow{\scriptstyle f_*} & & \downarrow \\
\mathfrak{M}_A^{\alpha+1} D & \longrightarrow & \mathfrak{M}_A^{\alpha+1} E
\end{array}
$$

is a homotopy pushout in $\mathcal{M}ap^{\alpha+1}$ (cf. [Mat]). Because we have a weak equivalence $\widehat{\mathcal{C}}_A^{\alpha+1} \to \mathcal{C}_A^{\alpha+1}$ (in $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$), using Lemma 1.19 we see that (6.4) is a homotopy pushout, too. $\qquad\square$

6.8. **Definition.** As in §3.1, we define an $\hat{\mathfrak{T}}_A^\lambda$-*algebra* to be a discrete λ-extended A-mapping algebra $\mathfrak{X}$ equipped with a natural map $\widehat{h}_{\mathfrak{X}}^\lambda : \hat{\mathfrak{T}}_A^\lambda \mathfrak{X} \to \mathfrak{X}$ such that the diagram corresponding to (3.2) commutes *up to homotopy* – that is:

(6.9)
$$
\widehat{h}_{\mathfrak{X}}^\lambda \circ (\widehat{\mathfrak{M}}_A^\lambda)_\delta \varepsilon_{\tilde{L}_A \rho \mathfrak{X}} \;\sim\; \widehat{h}_{\mathfrak{X}}^\lambda \circ (\hat{\mathfrak{T}}_A^\lambda)_\delta (\widehat{h}_{\mathfrak{X}}^\lambda) \;.
$$

6.10. *Remark.* Note that (6.9) holds automatically for any lax λ-extended A-mapping algebra $\hat{\mathfrak{X}}$, since $\widehat{h}_{\mathfrak{X}}^\lambda$ encodes the (lax) pre-compositions with maps from $\mathcal{C}_A^\lambda$, as in Proposition 3.7. Thus lax λ-extended A-mapping algebras are in particular $\hat{\mathfrak{T}}_A^\lambda$-algebras. However, the analogue of Remark 3.8 does not hold here, since (6.9) does not guarantee higher homotopy commutativity.

6.11. **Proposition.** *Given two weakly equivalent fibrant simplicial sets X and Y, any lax A-mapping algebra structure $\hat{\mathfrak{X}}$ on X determines a lax A-mapping algebra structure $\hat{\mathfrak{Y}}$ on Y, unique up to weak equivalence.*

Proof. Let $\mathcal{C}_X$ denote the $(\mathcal{S}, \mathcal{O})$-category of §1.17, where $\mathcal{O}$ is the object set of $\mathcal{C}_A^\lambda$ (since X is fibrant, we may assume that $\mathcal{C}_X$ is fibrant in $(\mathcal{S}, \mathcal{O})$-$\mathcal{C}at$). By Lemma 1.19, the lax A-mapping algebra structure $\hat{\mathfrak{X}}$ on X is determined by a map of $(\mathcal{S}, \mathcal{O})$-categories $f : \widehat{\mathcal{C}}_A^\lambda \to \mathcal{C}_X$.

Since X and Y are weak equivalent Kan complexes, there is a homotopy equivalence $h : Y \to X$ by [Q, I, §1, Theorem 1]. Factoring h as a cofibration followed by a fibration and using standard lifting properties, we obtain a diagram of weak equivalences:

(6.12)
$$
\begin{array}{ccccc}
& & Z & & \\
& {\scriptstyle q}\nearrow \;\; {\scriptstyle i}\nearrow & & {\scriptstyle j}\nwarrow \;\; {\scriptstyle p}\searrow & \\
Y & & \xrightarrow{\;\;h\;\;} & & X
\end{array}
$$

with p a fibration, i a cofibration, $p \circ i = h$, $q \circ i = \mathrm{Id}_X$, and $p \circ j = \mathrm{Id}_Y$ (so j is a cofibration, too).

If we use the usual functorial constructions of products, (homotopy) pullbacks and sequential limits in $\mathcal{S}_*$, we may assume that they preserve objectwise fibrations and cofibrations, as well as weak equivalences. Thus if $L_I(X)$ is one of the objects of $\mathcal{C}_X$, obtained from X by iterated homotopy limits as above, we have a trivial cofibration $L_I(j) : L_I(X) \to L_I(Z)$, inducing a commuting diagram of weak equivalences on

mapping spaces

$$(6.13) \quad \begin{array}{ccc} \operatorname{map}_*(L_I(X), X) & \xrightarrow[\simeq]{L_I(p)^*} \operatorname{map}_*(L_I(Z), X) & \xrightarrow[\simeq]{j_*} \operatorname{map}_*(L_I(Z), Z) \\ \downarrow{\scriptstyle\operatorname{Id}} & & {\scriptstyle\simeq}\Bigg\downarrow{\scriptstyle L_I(j)^*} \\ \operatorname{map}_*(L_I(X), X) & \xleftarrow[p_*]{\simeq} \operatorname{map}_*(L_I(X), Z) & \end{array}$$

with $L(i)^*$ and p_* fibrations (by [Q, II, §2, SM7]). By (1.18), the trivial fibrations $L(i)^* \circ p_* : \operatorname{map}_*(L_I(Z), Z) \to \operatorname{map}_*(L_I(X), X)$ fit together to define a trivial fibration $\Phi : \mathcal{C}_Z \to \mathcal{C}_X$ in $(\mathcal{S}, \mathcal{O})\text{-}\mathcal{C}at$ (with section $\Psi : \mathcal{C}_X \to \mathcal{C}_Z$).

Since $\widehat{\mathcal{C}}_A^\lambda$ is cofibrant, we have a lifting in $(\mathcal{S}, \mathcal{O})\text{-}\mathcal{C}at$:

$$(6.14) \quad \begin{array}{ccc} * & \longrightarrow & \mathcal{C}_Z \\ \downarrow & {\scriptstyle\hat{f}}\nearrow & {\scriptstyle\simeq}\Big\downarrow{\scriptstyle\Phi} \\ \widehat{\mathcal{C}}_A^\lambda & \xrightarrow{f} & \mathcal{C}_X \end{array}$$

and thus a lax A-mapping algebra structure on Z (corresponding to $\hat{f}$ by Lemma 1.19).

Similarly, the left hand side of (6.12) yields weak equivalences $L(i)_* \circ q^* : \operatorname{map}_*(L_I(Z), Z) \to \operatorname{map}_*(L_I(Y), Y)$, which fit together to define a weak equivalence of $(\mathcal{S}, \mathcal{O})$-categories $\Xi : \mathcal{C}_Z \to \mathcal{C}_Y$. Composing this with $\hat{f}$ yields a map $\Xi \circ \hat{f} : \widehat{\mathcal{C}}_A^\lambda \to \mathcal{C}_Y$, and thus a lax A-mapping algebra structure $\hat{\mathfrak{Y}}$ on Y. Since Ξ and Φ are weak equivalences, $\hat{\mathfrak{Y}}$ is unique up to weak equivalence. $\qquad\square$

6.15. The relaxed colimit construction. We now show how the construction of Section 5 can be applied to a lax A-mapping algebra structure $\hat{\mathfrak{X}}$ on mapping space $X = \operatorname{map}(A, Y)$:

In the notation of §4.1, for any ordinal $\alpha < \lambda$, at the α-th stage of the induction we assume that we have constructed the approximations $Y^{(\beta)} \in \mathcal{F}^{(\lambda)}$, $(\beta \leq \alpha)$, starting with $Y^{(0)} := * \in \mathcal{F}^{(\lambda)}$ as the 0-th approximation to Y.

At stage α of the induction we assume that we have a map of lax A-mapping algebras $\mathfrak{f}^{(\beta)} : \widehat{\mathfrak{M}}_A^\lambda Y^{(\beta)} \to \hat{\mathfrak{X}}$ and a map $\tilde{p}^{(\beta)} : \tilde{L}_A \rho \hat{\mathfrak{X}} \to \widetilde{Y}^{(\beta)}$ in $\mathcal{C}$ for each $\beta < \alpha$, as well as a cofibration $i^{(\beta)} : Y^{(\beta)} \to Y^{(\beta+1)}$ in $\mathcal{C}$ such that

$$(6.16) \qquad \mathfrak{f}^{(\beta+1)} \circ \widehat{\mathfrak{M}}_A^\lambda i^{(\beta)} ~\sim~ \mathfrak{f}^{(\beta)} \quad \text{and} \quad \mathfrak{f}^{(\beta+1)} \circ \widehat{\mathfrak{M}}_A^\lambda p^{(\beta+1)} ~\sim~ \widehat{h}_{\hat{\mathfrak{X}}}^\lambda$$

for each $\beta < \alpha$ (homotopic, not strictly equal). By Remark 6.10, we have an $\hat{\mathfrak{T}}_A^\lambda$-algebra structure $\widehat{h}_{\hat{\mathfrak{X}}}^\lambda$ on $\hat{\mathfrak{X}}$.

The construction of $Y^{(\alpha+1)}$ is identical to that of §4.7, with $\mathfrak{M}_A^\alpha$ and $\widetilde{\mathfrak{T}}^\alpha$ replaced throughout by $\widehat{\mathfrak{M}}_A^\lambda$ and $\hat{\mathfrak{T}}_A^\lambda$, respectively.

Again $Y^{(\alpha+1)}$ is defined by the analogue of (4.10):

$$(6.17) \quad \begin{array}{ccc} L_A Y^{(\alpha)} = \tilde{L}_A \rho \widehat{\mathfrak{M}}_A^\lambda Y^{(\alpha)} & \xhookrightarrow{\widetilde{\mathfrak{f}^{(\alpha)}}} & \widetilde{L_A \mathfrak{X}} \\ \downarrow{\scriptstyle \varepsilon_{Y^{(\alpha)}}} & & \downarrow{\scriptstyle p^{(\alpha+1)}} \\ Y^{(\alpha)} & \xhookrightarrow[i^{(\alpha)}]{\quad\boxed{\text{PO}}\quad} & \widetilde{Y}^{(\alpha+1)} \end{array}$$

in $\mathcal{C}$, which is a (homotopy) pushout of objects in $\mathcal{F}^{(\alpha)}$, so it is in $\mathcal{F}^{(\alpha+1)}$, and thus in $\mathcal{F}^{(\lambda)}$, too. Applying $\widehat{\mathfrak{M}}_A^\lambda$ to (6.17) thus yields a homotopy pushout diagram in $\widehat{\mathcal{M}ap_A^\lambda}$, by Lemma 6.5.

Moreover, the inner square in the following diagram commutes up to homotopy in $\widehat{\mathcal{M}ap_A^\lambda}$:

(6.18)

$$
\begin{array}{ccc}
\widehat{\mathfrak{M}}_A^\lambda L_A Y^{(\alpha)} & \xrightarrow{\;\widehat{\mathfrak{M}}_A^\lambda \tilde{L}_A \rho \mathfrak{f}^{(\alpha)}\;} & \widehat{\mathfrak{M}}_A^\lambda \tilde{L}_A \rho \hat{\mathfrak{X}} \\
{\scriptstyle \widehat{\mathfrak{M}}_A^\lambda \tilde{\varepsilon}_{Y^{(\alpha)}}}\downarrow & & \downarrow{\scriptstyle \widehat{\mathfrak{M}}_A^\lambda p^{(\alpha+1)}} \\
\widehat{\mathfrak{M}}_A^\lambda \tilde{Y}^{(\alpha)} & \xrightarrow{\;\widehat{\mathfrak{M}}_A^\lambda i^{(\alpha)}\;} & \widehat{\mathfrak{M}}_A^\lambda Y^{(\alpha+1)}
\end{array}
$$

with maps $h_{\hat{\mathfrak{X}}}^\lambda$, $\mathfrak{f}^{(\alpha+1)}$, $\mathfrak{f}^{(\alpha)}$ to $\hat{\mathfrak{X}}$.

by the induction hypothesis.

Since the upper left corner of (6.18) consists of free lax A-mapping algebras, by Lemma 6.4 we have a dotted map (of lax A-mapping algebras) making the full diagram (6.18) commute up to homotopy. This completes the induction step by defining $\mathfrak{f}^{(\alpha+1)}$.

If α is a limit ordinal, we let $Y^{(\alpha)} := \operatorname{colim}_{\beta<\alpha} Y^{(\beta)}$, as before.

The proof of Theorem 5.1 readily adapts to yield:

6.19. **Theorem.** *Let A be any λ-small object in a pointed simplicial model category $\mathcal{C}$, and X a pointed simplicial set equipped with a lax λ-extended A-mapping algebra structure $\hat{\mathfrak{X}}$. If we construct $Y := \tilde{Y}^{(\lambda)}$, as in §6.15, then the map of lax A-mapping algebras $\mathfrak{f}^{(\lambda)} : \widehat{\mathfrak{M}}_A^\lambda Y \to \hat{\mathfrak{X}}$ defined there induces a weak equivalence $X \simeq \operatorname{map}_{\mathcal{C}}(A,Y)$.*

6.20. *Remark.* The procedure described in Section 5 for realizing a *strict A-mapping algebra* $\mathfrak{X}$ can be simplified when $\mathcal{C} = \mathcal{T}_*$ for certain choices of A – e.g., when A is a wedge of (possibly localized) spheres. In this case we can use the Stover resolution for $\mathfrak{X}$, obtained by iterating the monad $\mathfrak{T}_A^\delta$ and applying $\tilde{L}_A$ to the resulting simplicial A-mapping algebra $\mathfrak{V}_\bullet$, with the missing face maps provided by the $\mathfrak{T}_A^\delta$-algebra structure $\varepsilon_{\mathfrak{X}} : \mathfrak{T}_A^\delta \mathfrak{X} \to \mathfrak{X}$ (see [BB1]). The ingredients allowing this simplified approach are:

(a) The fact that the loop space functor Ω commutes with the realization of simplicial spaces (which are connected CW complexes in each dimension) – see [May, Theorem 12.3].

(b) The fact that $- \otimes R$ is exact when R is a subring of the rationals.

(c) The fact that when $A = A_0 \vee A_1$, an A-mapping algebra encodes also the A_0- and A_1-mapping algebra structures.

6.21. **Lax Stover resolutions.** The usual Stover construction is not homotopy invariant, but it has a lax version which can be described as follows: assume that A is λ-compact, so that $\operatorname{map}_*(A,-)$ commutes with sequential colimits of length λ, and let Θ_A be the smallest subset of $\operatorname{Obj} \mathcal{C}$ containing A and closed under

suspensions and coproducts of cardinality less than λ. This suffices to define the comonad $L_A := \tilde{L}_A \circ \rho_0 \mathfrak{M}_A^\delta$ on $\mathcal{C}$, as well as the monad $\mathfrak{T}_A := \rho_0 \mathfrak{M}_A^\delta \circ \tilde{L}_A$ on the category $\mathcal{M}ap_A^\delta$ of discrete A-mapping algebras.

Let $\mathcal{C}_A$ be the full sub-simplicial category of $\mathcal{C}$ with objects $\mathbf{\Theta}_A$, , and $\widehat{\mathcal{C}}_A = \mathfrak{F}\pi_0(\mathcal{C}_A)$ the free simplicial resolution of its homotopy category. Given a discrete A-mapping algebra $\mathfrak{X}$ with a lax A-mapping algebra structure $\hat{\mathfrak{x}} : \widehat{\mathcal{C}}_A \to \mathcal{S}_*^{\mathrm{Kan}}$ on $X = \mathfrak{X}\{A\}$, we obtain a lax $\mathfrak{T}_A$-algebra structure $\varepsilon_{\mathfrak{x}} : \mathfrak{T}_A \mathfrak{X} \to \mathfrak{X}$ on $\mathfrak{X}$, so that we have a homotopy

$$(6.22) \qquad H : \varepsilon_{\mathfrak{x}} \circ \mathfrak{M}_A^\delta(\varepsilon_{\tilde{L}_A \rho \mathfrak{x}}) \xrightarrow{\sim} \varepsilon_{\mathfrak{x}} \circ \mathfrak{T}_A(\varepsilon_{\mathfrak{x}})$$

as in (6.9).

If we set $W_n := \tilde{L}_A \mathfrak{T}_A^n \rho \mathfrak{X}$ for each $n \geq 0$, the monad $\mathfrak{T}_A$ defines degeneracies and face maps making this into a simplicial object over $\mathcal{C}$, except for the *last* face map $d_n : W_n \to W_{n-1}$, which we define to be $\tilde{L}_A \mathfrak{T}_A^n \varepsilon_{\mathfrak{x}}$ (for $n \geq 0$). All the simplicial identities hold on the nose, except those involving the last face map, which hold up to homotopy (with the homotopy induced by H). Thus for example $d_0 d_1 \sim d_0 d_0$ is just $\tilde{L}_A$ applied to (6.22). However, in order to get higher homotopies, we need the full lax A-mapping algebra structure on $\hat{\mathfrak{x}}$, which make $\varepsilon_{\mathfrak{x}} : \mathfrak{T}_A \mathfrak{X} \to \mathfrak{X}$ into an ∞-$\mathfrak{T}_A$-algebra structure on $\mathfrak{X}$. With this extra structure, $W_\bullet$ becomes an ∞-homotopy commutative simplicial space, which can then be rectified to a strict simplicial space $V_\bullet$, using [BV, Theorem IV.4.37] or [DKSm, Theorem 2.4] Applying the Bousfield-Friedlander spectral sequence (see [BF, Theorem B.5]), we then deduce that the geometric realization $\|V_\bullet\|$ of $V_\bullet$ realizes the original A-mapping algebra $\mathfrak{X}$.

APPENDIX: A DOLD-LASHOF TYPE APPROACH

Here we sketch an alternative approach to recovering $\mathrm{CW}_A Y$ from $\mathrm{map}(A, X)$ by successive approximations in the spirit of the Dold-Lashof "projective spaces" $P^n \Omega Y$ for loop spaces (see [DL, F, Sta2]. Our construction is based directly on that of Dror-Farjoun for $\mathrm{CW}_A Y$ (in [DF, §2B]). The setting is still that of §0.4.

7.1. Mapping spaces and the monad T_A. For fixed cofibrant $A \in \mathcal{C}$, we define a functor $M_A : \mathcal{C} \to \mathcal{S}_*^{\mathrm{Kan}}$ by $M_A X := \mathrm{map}_*(A, X)$. This has a left adjoint $F_A : \mathcal{S}_* \to \mathcal{C}$ defined $F_A K := A \ltimes K := A \otimes K / A \otimes *$, when K is pointed (compare (2.3), where the other half-smash was used to define $\otimes$ for $\mathcal{C} = \mathcal{S}_*$). This is actually an enriched adjunction (cf. [Bor, §6.7]) – i.e., there are natural isomorphisms of (pointed) simplicial sets

$$\mathrm{map}_\mathcal{C}(F_A K, X) = \mathrm{map}_\mathcal{C}(A \ltimes K, X) \cong \mathrm{map}_{\mathcal{S}_*}(K, \mathrm{map}_\mathcal{C}(A, X)) = \mathrm{map}_{\mathcal{S}_*}(K, M_A X) \ .$$

This defines a monad $T_A := M_A F_A : \mathcal{S}_* \to \mathcal{S}_*$, equipped with natural transformations $\eta : \mathrm{Id} \to T_A$ (adjoint to Id_{F_A}) and $\mu : T_A T_A \to T_A$ (which is M_A applied to the adjoint of Id_{T_A}), satisfying the usual identities. A T_A-*algebra* is a pointed simplicial set $X \in \mathcal{S}_*$ equipped with a splitting $\varepsilon_X : T_A X \to X$ for

$\eta_X : X \to T_A X$, such that:

(7.2)

$$
\begin{array}{ccc}
T_A T_A X & \xrightarrow{\mu_X} & T_A X \\
{\scriptstyle T_A(\varepsilon_X)}\downarrow & & \downarrow{\scriptstyle \varepsilon_X} \\
T_A X & \xrightarrow{\varepsilon_X} & X
\end{array}
$$

commutes (cf. [Bor, §4.1].

Note that when $X = M_A Y$ for some $Y \in \mathcal{C}$, we may take $\varepsilon_{M_A Y} : M_A F_A M_A Y \to M_A Y$ to be $M_A(\widetilde{\mathrm{Id}_{M_A Y}})$, in which case (7.2) is satisfied automatically. Thus every A-mapping space is in particular an algebra over T_A. We wish to find conditions for the converse to hold.

7.3. **Reconstructing the target.** Let A be a fixed object in a pointed simplicial model category $\mathcal{C}$. For any $X = \mathrm{map}_{\mathcal{C}}(A, Y)$, we describe a (transfinite) inductive procedure for recovering Y from X (up to A-equivalence), together with its T_A-algebra structure. This procedure is essentially Dror-Farjoun's construction of the CW_A-approximation of Y (cf. [DF, §2B]), modified to take into account the fact that Y is not actually given.

We start with $Z^{(0)} := *$ as the 0-th approximation to Y in $\mathcal{C}$. For any ordinal $\alpha < \lambda$, at the α-th stage of the induction we assume that for each $\beta \le \alpha$, we have constructed $Z^{(\beta)} \in \mathcal{C}$, equipped with maps of T_A-algebras $\mathfrak{f}^{(\beta)} : M_A Z^{(\beta)} \to X$ as well as maps $p^{(\beta)} : F_A X \to Z^{(\beta)}$ in $\mathcal{C}$, together with an injective map of simplicial sets $i^{(\beta)} : Z^{(\beta)} \to Z^{(\beta+1)}$ such that

(7.4) $\qquad \mathfrak{f}^{(\beta+1)} \circ M_A i^{(\beta)} = \mathfrak{f}^{(\beta)} \quad$ and $\quad \mathfrak{f}^{(\beta+1)} \circ M_A p^{(\beta+1)} = \varepsilon_X$

for each $\beta < \alpha$.

If α is a limit ordinal, we let $Z^{(\alpha)} := \mathrm{colim}_{\beta < \alpha} Z^{(\beta)}$, with $\mathfrak{f}^{(\alpha)} := \mathrm{colim}_{\beta < \alpha} \mathfrak{f}^{(\beta)}$.

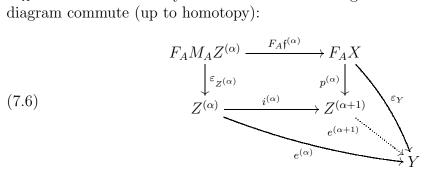We now define $Z^{(\alpha+1)}$ by the homotopy pushout diagram

(7.5)

$$
\begin{array}{ccc}
F_A M_A Z^{(\alpha)} & \xrightarrow{F_A \mathfrak{f}^{(\alpha)}} & F_A X \\
{\scriptstyle \varepsilon_{Z^{(\alpha)}}}\downarrow & \boxed{\mathrm{PO}} & \downarrow{\scriptstyle p^{(\alpha+1)}} \\
Z^{(\alpha)} & \xrightarrow{i^{(\alpha)}} & Z^{(\alpha+1)}
\end{array}
$$

in $\mathcal{C}$.

Note that *if* $X = M_A Y$, and we assume by induction that $\mathfrak{f}^{(\alpha)} : M_A Z^{(\alpha)} \to M_A Y = X$ is induced by $e^{(\alpha)} : Z^{(\alpha)} \to Y$ making the outer square in the following diagram commute (up to homotopy):

(7.6)

$$
\begin{array}{ccc}
F_A M_A Z^{(\alpha)} & \xrightarrow{F_A \mathfrak{f}^{(\alpha)}} & F_A X \\
{\scriptstyle \varepsilon_{Z^{(\alpha)}}}\downarrow & & \downarrow{\scriptstyle p^{(\alpha)}} \\
Z^{(\alpha)} & \xrightarrow{i^{(\alpha)}} & Z^{(\alpha+1)}
\end{array}
$$

then the pushout property guarantees the existence of map $e^{(\alpha+1)} : Z^{(\alpha+1)} \to Y$, unique up to homotopy, such that (7.4) holds for $\mathfrak{f}^{(\alpha+1)} := M_A e^{(\alpha+1)}$.

Applying M_A to (7.5), we obtain a commuting square in $\mathcal{S}_*$:

$$
(7.7) \qquad
\begin{array}{ccc}
T_A M_A Z^{(\alpha)} & \xrightarrow{\;T_A \mathfrak{f}^{(\alpha)}\;} & T_A X \\
{\scriptstyle M_A \varepsilon_{Z^{(\alpha)}}} \downarrow & & \downarrow {\scriptstyle M_A p^{(\alpha)}} \\
M_A Z^{(\alpha)} & \xrightarrow[\;M_A i^{(\alpha)}\;]{} & M_A Z^{(\alpha+1)}
\end{array}
$$

in $\mathcal{C}$ – though it is no longer homotopy cocartesian.

Nevertheless, if we assume that $\mathfrak{f}^{(\alpha)}$ is a map of T_A-algebras, the following square commutes:

$$
\begin{array}{ccc}
T_A M_A Z^{(\alpha)} & \xrightarrow{\;T_A \mathfrak{f}^{(\alpha)}\;} & T_A X \\
{\scriptstyle \varepsilon_{M_A Y} = M_A \varepsilon_{Z^{(\alpha)}}} \downarrow & & \downarrow {\scriptstyle \varepsilon_X} \\
M_A Z^{(\alpha)} & \xrightarrow[\;\mathfrak{f}^{(\alpha)}\;]{} & X
\end{array}
$$

Thus it makes sense to look for a dotted map (of T_A-algebras) making the following diagram commute:

$$
(7.8) \qquad
\begin{array}{ccc}
& T_A M_A Z^{(\alpha)} \xrightarrow{\;T_A \mathfrak{f}^{(\alpha)}\;} T_A X & \\
{\scriptstyle \varepsilon_{M_A Y} = M_A \varepsilon_{Z^{(\alpha)}}} \downarrow & \quad {\scriptstyle p^{(\alpha)}} \downarrow \qquad \searrow {\scriptstyle \varepsilon_X} & \\
M_A Z^{(\alpha)} \xrightarrow{\;M_A i^{(\alpha)}\;} M_A Z^{(\alpha+1)} & & \\
& \mathfrak{f}^{(\alpha+1)} \qquad & \\
& \searrow_{\mathfrak{f}^{(\alpha)}} \qquad X &
\end{array}
$$

7.9. **Definition.** Given a T_A-algebra X and a sequence of objects $Z^{(\beta)}$ equipped with maps $\mathfrak{f}^{(\beta)}$, $p^{(\beta)}$, and $i^{(\beta)}$ satisfying (7.4) for $\beta \le \alpha$, the $(\alpha + 1)$-*st lifting* for this sequence is any map of T_A-algebras $\mathfrak{f}^{(\alpha+1)} : M_A Z^{(\alpha+1)} \to X$ satisfying (7.4) for $\alpha + 1$, where $Z^{(\alpha+1)}$, $p^{(\alpha+1)}$, and $i^{(\alpha)}$ are defined by (7.5).

7.10. **Theorem.** *Let $A \in \mathcal{S}_*$ be a fixed simplicial set which is λ-small for $\lambda = \lambda(A)$ (Definition 1.12). If X is a T_A-algebra and $Z^{(\alpha)} \in \mathcal{C}$ $(\alpha \le \lambda)$ is a sequence of objects in $\mathcal{C}$ as above having α-liftings $\mathfrak{f}^{(\alpha)} : M_A Z^{(\alpha)} \to X$ for each $\alpha < \lambda$, then X is realizable as an A-mapping space $X \simeq \mathrm{map}(A, Y)$ for $Y := \mathrm{hocolim}_{\alpha < \lambda} Z^{(\alpha)}$.*

Proof. See [DF, §2 Proposition B.1], which is based on Bousfield's small object argument (cf. [Bou]). $\qquad \square$

References

[Ba]   B. Badzioch,"Algebraic theories in homotopy theory", *Ann. Math. (2)* **155** (2002), pp. 895-913.

[BD]   B. Badzioch & W. Dorabiała, "A note on localizations of mapping spaces", *Israel J. Math.* **177** (2010), pp. 441-444.

[Bl]   D. Blanc, "Mapping spaces and M-CW complexes", *For. Math.* **9** (1997), pp. 367-382.

[BB1]   H.-J. Baues & D. Blanc, "Comparing cohomology obstructions", *J. Pure & Appl. Alg.* **215** (2011), pp. 1420-1439.

[BB2]   H.-J. Baues & D. Blanc, "Stems and spectral sequences", *Alg. & Geom. Top.* **10** (2010), pp. 2061-2078.

[BV]   J.M. Boardman & R.M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Springer-Verlag *Lec. Notes Math.* **347**, Berlin-New York, 1973.

[Bor]   F. Borceux, *Handbook of Categorical Algebra, Vol. 2: Categories and Structures*, Encyc. Math. & its Appl. **51**, Cambridge U. Press, Cambridge, UK, 1994.

[Bou]   A.K. Bousfield, "The localization of spaces with respect to homology", *Topology* **14** (1975), pp. 135-150.

[BF]   A.K. Bousfield & E.M. Friedlander, "Homotopy theory of Γ-spaces, spectra, and bisimplicial sets", in M.G. Barratt & M.E. Mahowald, eds., *Geometric Applications of Homotopy Theory, II* Springer-Verlag *Lec. Notes Math.* **658**, Berlin-New York, 1978, 80–130

[CDI]   W. Chachólski, W.G. Dwyer, & M. Intermont, "The A-complication of spaces", *J. Lond. Math. Soc. (2)* **65** (2002), pp. 204-222.

[C]   P.V.Z. Cobb, "P_n-spaces and n-fold loop spaces", *Bull. AMS* **80** (1974), pp. 910-914.

[DF]   E. Dror-Farjoun, *Cellular spaces, null spaces, and homotopy localization*, Springer-Verlag *Lec. Notes Math.* **1622**, Berlin-New York, 1995.

[DL]   A. Dold & R.K. Lashof, "Principal quasifibrations and fibre homotopy equivalence of bundles", *Illinois J. Math.* **3** (1959), pp. 285-305.

[DK]   W.G. Dwyer & D.M. Kan, "Singular functors and realization functors", *Proc. Kon. Ned. Akad. Wet.* **2** (1984), pp. 147-153,

[DKSm]   W.G. Dwyer, D.M. Kan, & J.H. Smith, "Homotopy commutative diagrams and their realizations", *J. Pure & Appl. Alg.*, **57** (1989), pp. 5-24.

[DKSt1] W.G. Dwyer, D.M. Kan, & C.R. Stover, "An E^2 model category structure for pointed simplicial spaces", *J. Pure & Appl. Alg.* **90** (1993), pp. 137-152.

[E]   C. Ehresmann, "Esquisses et types des structures algébriques", *Bul. Inst. Politehn. Iaşi (N.S.)* **14** (1968), pp. 1-14.

[FT]   Y. Félix & D. Tanré, "H-space structure on pointed mapping spaces"", *Alg. & Geom. Top.* **5** (2005), pp. 713-724.

[F]   M. Fuchs, "A modified Dold-Lashof construction that does classify H-principal fibrations", *Math. Ann.* **192** (1971), pp. 328-340.

[Hi]   P.S. Hirschhorn, *Model Categories and their Localizations*, Math. Surveys & Monographs **99**, AMS, Providence, RI, 2002.

[Ho]   M.A. Hovey, *Model Categories*, Math. Surveys & Monographs **63**, AMS, Providence, RI, 1998.

[IS]   M. Intermont & J.A. Strom, "Homotopical complexity and good spaces", *Trans. AMS* **359** (2007), pp. 687-700.

[K]   G.M. Kelly, *Basic concepts of enriched category theory*, Cambridge U. Press, Cambridge, UK, 1982.

[McL]   S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag *Grad. Texts in Math.* **5**, Berlin-New York, 1971.

[Mat]   M. Mather, "Pull-backs in homotopy theory", *Can. J. Math.* **28** (1976), pp. 225-263.

[May]   J.P. May, *The Geometry of Iterated Loop Spaces*, Springer-Verlag *Lec. Notes Math.* **271**, Berlin-New York, 1972.

[Q]   D.G. Quillen, *Homotopical Algebra*, Springer-Verlag *Lec. Notes Math.* **20**, Berlin-New York, 1963.

[Sm]   J.H. Smith, "Simplicial group models for $\Omega^n S^n X$", *Israel J. Math.* **66** (1989), pp. 330-350..

[Sta1]   J.D. Stasheff, "Homotopy associativity of H-spaces, I,II", *Trans. AMS* **108** (1963), pp. 275-292; 293-312.

[Sta2]   J.D. Stasheff, *H-spaces from a Homotopy Point of View*, Springer-Verlag *Lec. Notes Math.* **161**, Berlin-New York, 1970.

[Ste]   N.E. Steenrod, "A convenient category of topological spaces", *Michigan Math. J,* **14** (1967), pp. 133-152.

[Sto]   C.R. Stover, "A Van Kampen spectral sequence for higher homotopy groups", *Topology* **29** (1990), pp. 9-26

[Su1]   M. Sugawara, "On a condition that a space is an H-space", *Math. J. Okayama U.* **6** (1957), pp. 109-129.

[Su2]   M. Sugawara, "Note on H-spaces", *Proc. Acad. Jap.* **36** (1960), pp. 598-600.

[V]   R.M. Vogt, "Convenient categories of topological spaces for homotopy theory", *Arch. Math. (Basel)* **22** (1971), pp. 545-555.

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